Quantum (PHYS 290): Guide 9 Solutions

Fall 2025 v1.0

Solutions:

(1) From our work in class we have

$$\hat{L}_y = z\hat{p}_x - x\hat{p}_z$$

and

$$\hat{L}_z = x\hat{p}_y - y\hat{p}_x.$$

The commutator is then

$$\left[\hat{L}_y, \hat{L}_z\right] = \left[z\hat{p}_x - x\hat{p}_z, x\hat{p}_y - y\hat{p}_x\right].$$

Positions and their momenta do not commute so sleuthing out those terms we have

$$\left[\hat{L}_y, \hat{L}_z\right] = \left[z\hat{p}_x, x\hat{p}_y\right] + \left[x\hat{p}_z, y\hat{p}_x\right]$$

Now doing the commutator algebra,

$$\left[z\hat{p}_x, x\hat{p}_y\right] + \left[x\hat{p}_z, y\hat{p}_x\right] = z\left[\hat{p}_x, x\right]\hat{p}_y + y\left[x, \hat{p}_x\right]\hat{p}_z = \left(y\hat{p}_z - z\hat{p}_y\right)\left[x, \hat{p}_x\right].$$

The commutator $[x, \hat{p}_x] = i\hbar$ so

$$\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\left(y\hat{p}_z - z\hat{p}_y\right) = i\hbar\hat{L}_x$$

as expected. Neat! The other two in the set

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z,\ [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x,\ \text{and } [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y \tag{1}$$

work the same way. This can be done by repeating the steps above or simply permuting (y, z, x) (the computation we just did) to (z, x, y) and (x, y, z).

(2) For this commutation relation we need a form of $\hat{L}^2$. The easiest is in cartesian coordinates

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2.$$

Now,

$$[\hat{L}^2, \hat{L}_z] = [\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \hat{L}_z] = [\hat{L}_x^2 + \hat{L}_y^2, \hat{L}_z],$$

since any operator commutes with itself. From the identity $[AB, C] = A[B, C] + [A, C]B$ we have

$$[\hat{L}^2, \hat{L}_z] = \hat{L}_x[\hat{L}_x, \hat{L}_z] + [\hat{L}_x, \hat{L}_z]\hat{L}_x + \hat{L}_y[\hat{L}_y, \hat{L}_z] + [\hat{L}_y, \hat{L}_z]\hat{L}_y.$$

Using the commutators from equation 1, this becomes

$$\hat{L}_x(-i\hbar\hat{L}_y) + (-i\hbar\hat{L}_y)\hat{L}_x + \hat{L}_y(i\hbar\hat{L}_x) + (i\hbar\hat{L}_x)\hat{L}_y = 0$$

So

$$[\hat{L}^2, \hat{L}_z] = 0.$$

The other two relation $[\hat{L}^2, \hat{L}_y] = 0$ and $[\hat{L}^2, \hat{L}_x] = 0$, follow similarly; in spherical symmetry there is nothing distinguished about any one of the coordinate axes. Any pair of such eigenvalues, such as ℓ and m_z or ℓ and m_x, completely specify the angular states.

(3) Let's start with the commutator

$$\left[\hat{L}_z, \hat{L}_+\right] = \hbar \hat{L}_+.$$

Working from the definition of the raising operator

$$\begin{aligned}[\hat{L}_z, \hat{L}_+] &= [\hat{L}_z, \hat{L}_x + i\hat{L}_y] = [\hat{L}_z, \hat{L}_x] + i[\hat{L}_z, \hat{L}_y] \\ &= i\hbar\hat{L}_y + i(-i\hbar\hat{L}_x) = \hbar\hat{L}_+\end{aligned}$$

as desired. Similarly, from the definition of the lowering operator

$$\begin{aligned}[\hat{L}_z, \hat{L}_-] &= [\hat{L}_z, \hat{L}_x - i\hat{L}_y] = [\hat{L}_z, \hat{L}_x] - i[\hat{L}_z, \hat{L}_y] \\ &= i\hbar\hat{L}_y - i(-i\hbar\hat{L}_x) = -\hbar\hat{L}_-\end{aligned}$$

(4) Since the right hand side has no $\hat{L}_z$ operator , we need to commute the two operators and see what we get. So, from the last problem,

$$\left[\hat{L}_z, \hat{L}_-\right] = -\hbar\hat{L}_-, \text{ and } \hat{L}_z\hat{L}_- = \hat{L}_-\hat{L}_z - \hbar\hat{L}_-$$

so that

$$\hat{L}_z\left(\hat{L}_- \mid \psi\rangle\right) = \left(\hat{L}_-\hat{L}_z - \hbar\hat{L}_-\right) \mid \psi\rangle = \hbar(m-1)\left(\hat{L}_- \mid \psi\rangle\right).$$

This says that the m_z eigenvalue of the lowering operator acting on a state $\mid \psi\rangle$ with $m_z = m$ is one unit less; the m state is lowered by one unit.

(5) (Optional worth 1 extra pt.) The form of $\hat{L}_x$ and $\hat{L}_y$ in spherical coordinates. Here's a quick sketch of the solution. Following the hint and taking the cross products - I just used the right hand rule for these -

$$\hat{\mathbf{L}} = -i\hbar\left(\mathbf{u}_\theta\frac{\partial}{\partial\theta} + \mathbf{u}_\varphi\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

Substituting the form of these spherical unit vectors in cartesian coordinates yields

$$\hat{\mathbf{L}} = -i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\hat{\imath} - i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)\hat{\jmath} + \frac{\partial}{\partial\varphi}\hat{k}.$$

The first two terms are $\hat{L}_x$ and $\hat{L}_y$, respectively.

(6) The ladder operator $\hat{L}_+$ is

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y.$$

But in spherical coordinates $\hat{L}_x$ and $\hat{L}_y$ are

$$\hat{L}_x = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$$

and

$$\hat{L}_y = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right).$$

So

$$\hat{L}_+ = -i\hbar\left[(-\sin\varphi - i\cos\varphi)\frac{\partial}{\partial\theta} - \cot\theta(\cos\varphi + i\sin\varphi)\frac{\partial}{\partial\varphi}\right].$$

The trig terms in parenthesis are proportional to $e^{i\varphi}$ by Euler's formula. Collecting factors of i

$$\hat{L}_+ = \hbar\left[(\cos\varphi + i\sin\varphi)\frac{\partial}{\partial\theta} + i\cot\theta(\cos\varphi + i\sin\varphi)\frac{\partial}{\partial\varphi}\right]$$

or

$$\hat{L}_+ = \hbar e^{\pm i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

as hoped. The lowering case just has a propagating sign from the $-i\hat{L}_y$ term but otherwise the calculation is the same.

(7) The top level of the ladder is the state $Y_{\ell\,\ell}(\theta,\varphi)$. Lowering this gives

$$\hat{L}_- Y_{\ell\,\ell}(\theta,\varphi) \propto \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\left(\sin\theta e^{i\varphi}\right)^{\ell}$$

where I have left out the normalization of $Y_{\ell\ell}$. Acting with the derivatives gives

$$\hat{L}_- Y_{\ell\,\ell}(\theta,\varphi) \propto -\hbar\ell\sin^{\ell-1}\theta\cos\theta e^{i(\ell-1)\varphi} + i\hbar\frac{\cos\theta}{\sin\theta}\sin^{\ell}\theta(i\ell)e^{i(\ell-1)\varphi}$$

Combining terms gives

$$\hat{L}_- Y_{\ell\,\ell}(\theta,\varphi) \propto -2\hbar\ell\sin^{\ell-1}\theta\cos\theta e^{i(\ell-1)\varphi}.$$

For $\ell = 2$ this becomes

$$\hat{L}_- Y_{2\,2}(\theta,\varphi) \propto \sin\theta\cos\theta e^{i\varphi},$$

which is indeed proportional to the Y_{21} spherical harmonic in (6.48).

Using our result from class,

$$Y_{\ell\,\ell}(\theta,\varphi) \propto \left(\sin\theta e^{i\varphi}\right)^{\ell} \text{ when } \ell = 2 \text{ gives } Y_{2\,2}(\theta,\varphi) \propto \sin^2\theta e^{i2\varphi}$$

which matches what is in (6.47).

Lowering again on Y_{21} gives

$$\hat{L}_- Y_{2\,1}(\theta,\varphi) \propto \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\sin\theta\cos\theta e^{i\varphi}.$$

Computing the derivatives,

$$Y_{2\,0}(\theta,\varphi) \propto \hbar e^{-i\varphi}\left(-\cos^2\theta + \sin^2\theta - \frac{\cos\theta}{\sin\theta}\sin\theta\cos\theta\right)e^{i\varphi}$$

or using $\sin^2\theta = 1 - \cos^2\theta$,

$$Y_{2\,0}(\theta,\varphi) \propto 3\cos^2\theta - 1,$$

as expected from equation (6.49).

Finally lowering one last time,

$$Y_{2\,-1}(\theta,\varphi) \propto \hat{L}_- Y_{2\,0}(\theta,\varphi) \propto \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)(3\cos^2\theta - 1)$$

or

$$Y_{2\,-1}(\theta,\varphi) \propto \sin\theta\cos\theta e^{-i\varphi}$$

as in (6.48). Hooray! It all seems to work.

(8) Spinning NH_3:

(a) Using the angular momentum operators, the energy of Hamiltonian operator acting on a $|\,\ell\, m\rangle$ state is

$$\hat{H}\,|\,\ell\, m\rangle = \left(\frac{\hat{L}^2 - \hat{L}_x^2}{2I_1} + \frac{\hat{L}_z^2}{2I_3}\right)|\,\ell\, m\rangle = \left(\frac{\hbar^2\ell(\ell+1) - \hbar^2 m^2}{2I_1} + \frac{\hbar^2 m^2}{2I_3}\right)|\,\ell\, m\rangle$$

The two states are $|\,0\,0\rangle$ and $|\,1\,1\rangle$. From above the Hamiltonian acting on the first state is

$$\hat{H}\,|\,0\,0\rangle = 0, \text{ so } E_{00} = 0.$$

Acting on the second state

$$\hat{H}\,|\,1\,1\rangle = \left(\frac{2\hbar^2 - \hbar^2}{2I_1} + \frac{\hbar^2}{2I_3}\right)|\,1\,1\rangle = \frac{\hbar^2}{2}\left(\frac{1}{I_1} + \frac{1}{I_3}\right)|\,1\,1\rangle \equiv E_{11}\,|\,1\,1\rangle.$$

(BTW

$$\hat{H}\,|\,\psi\rangle = 0\frac{1}{\sqrt{2}}\,|\,0\,0\rangle + \frac{\hbar^2}{2\sqrt{2}}\left(\frac{1}{I_1} + \frac{1}{I_3}\right)|\,1\,1\rangle = \frac{\hbar^2}{2\sqrt{2}}\left(\frac{1}{I_1} + \frac{1}{I_3}\right)|\,1\,1\rangle.$$

This wavefunction is not an energy eigenstate.)

(b) For the time dependent form we just need to add the phases $e^{-iEt/\hbar}$ as we have done before. Thus,

$$|\,\Psi\rangle = \frac{1}{\sqrt{2}}\,|\,0\,0\rangle + \frac{e^{-iE_{11}t/\hbar}}{\sqrt{2}}\,|\,1\,1\rangle.$$

(c) The probability of the 0 eigenvalue is just the square of the 00 state amplitude, or $P = 1/2$. The expectation value is the energy value times the probability for that state

$$\langle E\rangle = E_{00}P(00) + E_{11}P(11) = \frac{E_{11}}{2} = \frac{\hbar^2}{4}\left(\frac{1}{I_1} + \frac{1}{I_3}\right)$$

(9) To check the commutation relation $[\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x$ we have two computations to do:

$$\hat{S}_y\hat{S}_z = \frac{\hbar^2}{4}\sigma_y\sigma_z = \frac{\hbar^2}{4}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$$

and

$$-\hat{S}_z\hat{S}_y = -\frac{\hbar^2}{4}\sigma_z\sigma_y = -\frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$$

So that

$$[\hat{S}_y, \hat{S}_z] = 2\cdot\frac{\hbar^2}{4}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} = i\hbar\cdot\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = i\hbar\hat{S}_x$$

as expected.