4/19/23

$$\mathcal{E}_1 = \frac{-d\Phi_{12}}{dt} = -M_{12}\frac{dI_2}{dt}$$

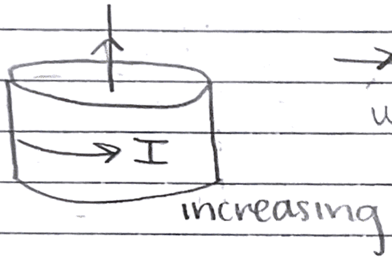
⇒ how are M_{12} and M_{21} related? (they are equal!)

Last time: Faraday's Law: 4/21/23

$$\underbrace{\int \vec{E}\cdot d\vec{s}}_{\mathcal{E}} = \frac{-d}{dt}\underbrace{\int \vec{B}\cdot d\vec{a}}_{\Phi_B}$$

Today: Thompson jumping rings: $I(t) = I_0\cos(\omega t)$

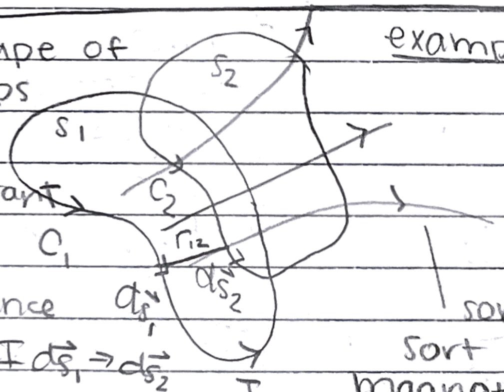
$f = 60\text{Hz}$



→ current causes ring to jump
w/ ring + slice → doesn't jump
w/copper ring → heavier but better flux = same sized jump as before

example: EMF in two loops

shape of loops is constant



r_{12}: distance btween $I d\vec{s}_1 \Rightarrow d\vec{s}_2$

some sort of magnetic flux activity

$$\mathcal{E}_2 = \frac{-d}{dt}\int_{S_2} B_1\cdot d\vec{a}_2$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

under certain conditions:

$$\nabla^2\vec{A} = \mu_0\vec{J} \quad \left(\nabla^2 V = -\frac{\rho}{\epsilon_0}\right)$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{I\,d\vec{s}}{r}$$

(by analogy)

$$\mathcal{E}_2 = \frac{-d}{dt}\int\vec{\nabla}\times\vec{A}_1\cdot d\vec{a}_2$$

stoke's theorem

$$= \frac{-d}{dt}\int_{C_2}\vec{A}_1\cdot d\vec{s}_2 = \frac{-d}{dt}\int_{C_2}\int_{C_1}\frac{\mu_0}{4\pi}\frac{I_1\,d\vec{s}_1\cdot d\vec{s}_2}{r_{12}}$$

now $\vec{A}_1 = \frac{\mu_0}{4\pi}\int_{C_1}\frac{I_1\,d\vec{s}_1}{r_{12}}$

current has no time dependance; shape of loops = constant

$$= \frac{-dI_1}{dt}\underbrace{\frac{\mu_0}{4\pi}\int_{C_1}\int_{C_2}\frac{d\vec{s}_1\cdot d\vec{s}_2}{r_{12}}}$$

geometry only: depends on shape = mutual inductance: M_{12}

★continued on next page★

4/21/23

$$\mathcal{E}_2 = -M_{12}\frac{dI_1}{dt}$$

⇒ now in terms of effect on other ring(1):

$$\mathcal{E}_1 = \frac{-d}{dt}\int_{S_1} B_2 \cdot d\vec{a}_1 = \frac{dI_2}{dt}\underbrace{\frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2}\frac{d\vec{S}_2 \cdot d\vec{S}_1}{r_{21}}}_{M_{21} = M_{12}}$$

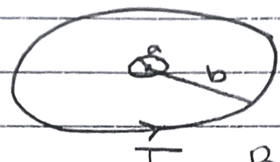
Therefore: $M = M_{12} = M_{21}$

→ doesn't matter!

given a certain configuration we can compute M, using soley geometry: <u>find flux + divide by I.</u>

<u>example:</u>

(little loop: rad a inside of a big loop: rad b)



magnetic field on axis of a loop is:

$$B_z = \frac{\mu_0 I b^2}{2(b^2+z^2)^{3/2}}$$

in loop a: $a \ll b$

→ if there are concentric circles: $B \approx \frac{\mu_0 I}{2b}$

$$\Phi = \int \vec{B}\cdot d\vec{a} = B\pi a^2$$ since $B \approx$ constant in loop a

$$B\pi a^2 = \frac{\mu_0 \pi I a^2}{2b}$$

$$M = \frac{\Phi}{I} = \frac{\mu_0 \pi \cancel{I} a^2}{2b\cancel{I}} = \boxed{\frac{\mu_0 \pi a^2}{2b}}$$

→ suppose instead of single loop a, "n" loops!

$$M = N\left(\frac{\mu_0 \pi a^2}{2b}\right)$$

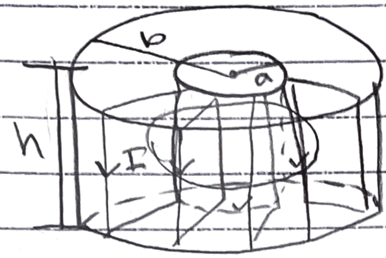
This works for one thing, too

⇒ self inductance: L

$$\mathcal{E} = \frac{-dI}{dt}L$$

4/21/23

example: toroid w/ N windings:



$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{encl}$$

amperean loop = circle around a w/ radius "r"

$$B\,2\pi r = \mu_0 I_{encl} = \mu_0 I N$$

$$\boxed{B = \frac{\mu_0 N I}{2\pi r}\,\hat{\phi}}$$

flux: $\Phi = \int \vec{B}\cdot d\vec{a}$

$$= \int B\,dy\,dr = \int dy \int B\,dr = h\int B\,dr$$

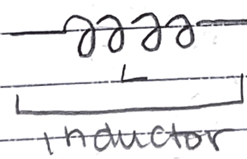
$$= \frac{h\mu_0 NI}{2\pi}\int_a^b \frac{1}{r}\,dr = \frac{h\mu_0 NI}{2\pi}\ln(r)\Big|_a^b$$

$$= \frac{h\mu_0 NI}{2\pi}\ln\left(\frac{b}{a}\right) \text{ for 1 loop}$$

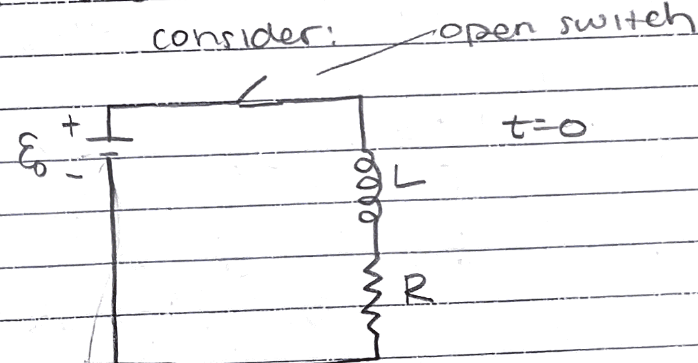
→ for all loops: $\Phi = N\left(\frac{h\mu_0 NI}{2\pi}\ln\left(\frac{b}{a}\right)\right) = \frac{h\mu_0 N^2 I}{2\pi}\ln\left(\frac{b}{a}\right)$

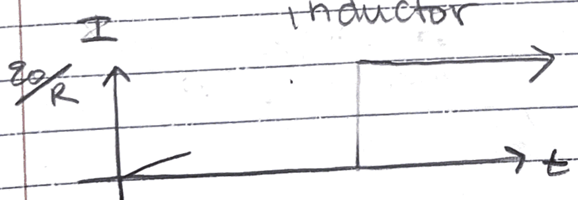
Therefore $L = \frac{\Phi}{I} = \frac{\mu_0 N^2 I h}{2\pi I}\ln\left(\frac{b}{a}\right) = \boxed{\frac{\mu_0 N^2 h \ln\left(\frac{b}{a}\right)}{2\pi}}$

In circuits, inductors are denoted



consider:





→ closing switch: $t>0$:

$I(t) =$

kirkoff loop rule:

$$\mathcal{E}_0 - L\frac{dI}{dt} - IR = 0 \quad \rightarrow$$

use diff eq:

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{\mathcal{E}_0}{L}$$

one-spatial, one-time dimension: 4/26/24

1+1 $\frac{\partial^2 f}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 f}{\partial t^2}$ $f(t,x)$

$\boxed{\text{wave-eq}}$

V : velocity (wave)
phase velocity

last time:

Maxwell's equations:

guass': $\nabla \cdot E = \rho/\epsilon_0$ $\vec{\nabla}\cdot\vec{B} = 0$

faraday: $\vec{\nabla}\times\vec{E} = \frac{-\partial\vec{B}}{\partial t}$ $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \overbrace{\underbrace{\mu_0\epsilon_0\frac{\partial E}{\partial t}}_{\vec{J_d}}}^{\text{Maxwell's correction}}$

today:

- applications of maxwell's eq.

displacement current $= \mu_0(\vec{J} + \vec{J_d})$

★math fact★: consider vector field: $\vec{V} = \vec{V}(x,y,z)$

(curl of curl)

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$\nabla^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2}$$

$$= \underbrace{\hat{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)}_{x} + \underbrace{\hat{j}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)}_{y} + \underbrace{\hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)}_{z}$$

$\vec{\nabla}\times(\vec{\nabla}\times\vec{V})$

$$= \hat{i}\left[\left(\frac{\partial}{\partial y}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\right) - \frac{\partial}{\partial z}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\right] + \ldots$$

(ignoring $\hat{j}$ + $\hat{k}$)

$$= \hat{i}\left(-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial}{\partial x}\frac{\partial V_y}{\partial y} + \frac{\partial}{\partial x}\frac{\partial V_z}{\partial z}\right) + \ldots$$

$$= -\nabla^2 V_x - \cancel{\frac{\partial^2 V_x}{\partial x^2}} + \cancel{\frac{\partial^2 V_x}{\partial x^2}} + \vec{\nabla}(\vec{\nabla}\cdot\vec{V})$$

$$\boxed{\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = -\nabla^2 V + \vec{\nabla}(\vec{\nabla}\cdot\vec{V})}$$

In vacuum: maxwell's equations:

$$\vec{\nabla}\cdot\vec{E}=0 \qquad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \nabla\times\vec{B}=\mu_0\epsilon_0\frac{\partial E}{\partial t}$$

↳ take curl of ampere:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=\mu_0\epsilon_0\,\vec{\nabla}\times\left(\frac{\partial\vec{E}}{\partial t}\right)$$

left-hand side → $-\nabla^2\vec{B}+\vec{\nabla}(\vec{\nabla}\cdot\vec{B})=-\nabla^2\vec{B}$

right-hand side → $\mu_0\epsilon_0\,\vec{\nabla}\times\left(\frac{\partial\vec{E}}{\partial t}\right)$

$$=\mu_0\epsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})=\mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\frac{\partial B}{\partial t}\right)$$

$$\therefore\ -\vec{\nabla}^2\vec{B}=-\mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

wave ♡ equation!

$$\vec{\nabla}^2\vec{B}=\mu_0\epsilon_0\frac{\partial^2 B^2}{\partial t^2}=\boxed{\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}}$$

↳ the wave travels at $\frac{1}{v^2}=\mu_0\epsilon_0 \Rightarrow v=\frac{1}{\sqrt{\mu_0\epsilon_0}}$

$\boxed{v=3\times10^8\,m/s}$

speed of light!

★ magnetic field has wave solutions and they travel at c!

↳ take curl of:

$$\vec{\nabla}(\vec{\nabla}\times\vec{E})=\vec{\nabla}\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

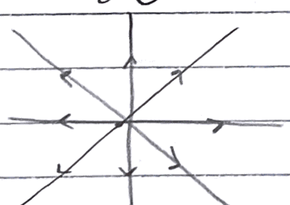
left-hand side → $-\nabla^2\vec{E}+\vec{\nabla}(\vec{\nabla}\cdot\vec{E})=-\nabla^2\vec{E}$

right-hand side → $\vec{\nabla}\left(-\frac{\partial\vec{B}}{\partial t}\right)=-\frac{\partial}{\partial t}\mu_0\epsilon_0\frac{\partial E}{\partial t}$

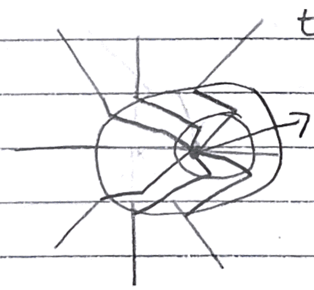
$$\therefore\ \boxed{\nabla^2\vec{E}=\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t^2}}$$

side: e-charges

t=0



t= 1ns



→ c

EM-wave

∃ kink in field lines

★ accelerated charge gives an E & M wave

→ next time:

$$E(t,\vec{r})=\vec{E}_0\cos(\vec{k}\cdot\vec{r}\pm\omega t)$$

last time: $\boxed{\text{light!}}$ 4/28/23

maxwell's equations in VAC:

$$\vec{\nabla}\cdot\vec{E}=0 \qquad \vec{\nabla}\cdot\vec{B}=0$$
$$\vec{\nabla}\times\vec{E}=\frac{-\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B}=\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

review from last class: $\mu_0\epsilon_0=\frac{1}{c^2}$

$$\Rightarrow \nabla^2 E=\frac{1}{c^2}\frac{\partial^2 E}{\partial t}$$

solutions? $\vec{E}(t,\vec{r})=\vec{E}_0\cos(\underbrace{\vec{k}\cdot\vec{r}}_{k_x x+k_y y+k_z z}\pm\omega t)$ | $\vec{B}(t,\vec{r})=B_0\cos(\vec{k}\cdot\vec{r}\pm\omega t)$

plane waves = good far from source

$$\Rightarrow \nabla^2\vec{B}=\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t}$$

- check wave-equation for e-field:

$$\frac{\partial\vec{E}}{\partial t}=\vec{E}_0\sin(\vec{k}\cdot\vec{r}-\omega t)\omega$$

$$\frac{\partial^2\vec{E}}{\partial t}=-\vec{E}_0\cos(\vec{k}\cdot\vec{r}-\omega t)\omega^2=-\omega^2 E$$

$$\nabla^2\vec{E}=\vec{\nabla}\cdot\vec{\nabla}\vec{E}=$$
$$\vec{\nabla}\vec{E}=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\vec{E}_0\cos(\vec{k}\cdot\vec{r}-\omega t)$$
$$=\vec{E}_0\underbrace{(\hat{x}\cdot k_x+\hat{y}k_y+\hat{z}k_z)}_{=\vec{k}}(-\sin(\vec{k}\cdot\vec{r}-\omega t))$$
$$=\vec{E}_0\vec{k}(-\sin(\vec{k}\cdot\vec{r}-\omega t))=\vec{k}\vec{E}$$

Therefore: $\nabla^2\vec{E}=-(\vec{k}\cdot\vec{k})\vec{E}_0\cos(\vec{k}\cdot\vec{r}-\omega t)=\boxed{-\vec{k}^2\vec{E}}$

$\longrightarrow$ every application of ∇ produces a $\vec{k}$

wave equation: $-k^2\vec{E}=\frac{-1}{c^2}\omega^2\vec{E}\Rightarrow\omega=c|\vec{k}|$

★recall: $|\vec{k}|=k=\frac{2\pi}{\lambda}$ — wave number; $\vec{k}$ = wave vector

verifying $\vec{\nabla}\times\vec{E}=\frac{\partial\vec{B}}{\partial t}$:

$$\frac{\partial\vec{B}}{\partial t}=\omega_0\vec{B}_0\sin(\vec{k}\cdot\vec{r}-\omega t)$$
$$\vec{\nabla}\times\vec{E}=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\times E_0\cos(\vec{k}\cdot\vec{r}-\omega t)$$
$$=-\vec{k}\times-\vec{E}_0\sin(\vec{k}\cdot\vec{r}-\omega t)$$

Faraday $\Rightarrow$ $-\vec{k}\times E_0\sin(\vec{k}\cdot\vec{r}-\omega t)=-\omega\vec{B}_0\sin(\vec{k}\cdot\vec{r}-\omega t)$

$$\vec{k}\times\vec{E}_0=\omega\vec{B}_0=c|\vec{k}|\vec{B}_0$$

from $\omega=|\vec{k}|c$

$$\cancel{|\vec{k}|}\hat{k}\times\vec{E}_0=c\cancel{|\vec{k}|}\vec{B}_0$$

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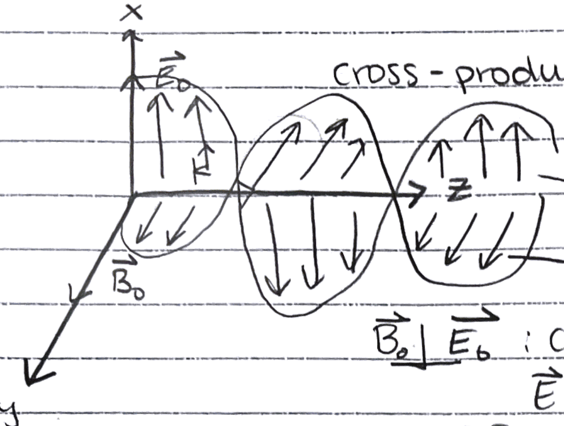

cross-product goal:

e-field + e-field vectors

b-field + b-field vectors

$\vec{B}_0 \perp \vec{E}_0$: changing $\vec{B}$ produces $\vec{E}$, changing $\vec{E}$ produces $\vec{B}$

faraday: $\nabla \times E = \frac{-\partial B}{\partial t}$ $\Rightarrow$ $\vec{\nabla} \cdot \vec{E} = 0$

$\oint \vec{E} \cdot d\vec{s} = \frac{-d}{dt} \int \vec{B} \cdot d\vec{a}$ $\quad$ $\vec{K} \cdot \vec{E} = 0$

$\vec{K} \cdot \vec{B} = 0$

★ Linear system, so we can superimpose solutions: $\vec{E}(t, \vec{r}) = E_0 \hat{z} \cos(kz - \omega t) + E_1 \hat{y} \sin(kz - \omega t)$

can change components + still have a solution!

- can also generate any wave from sines and cosines (FOURIER)

since: $\omega = |\vec{K}| c$ holds:

~~ultra violet + x-rays~~ + gamma rays ($1nm \rightarrow 10^{-11} m$)

~400nm (blue)

(visible)

~700nm (red)

(infra-red)

$\sim 10^{-3} m$

~~microwave~~

(1m) (microwave)

(radio-wave)

- in terms of energy units ≈ 1eV

E_0 is the polarization of the wave