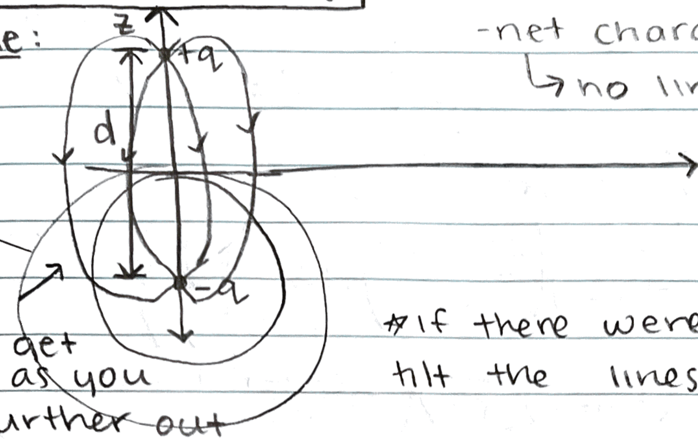
review: last time: * div in cylindrical co-ords — 2/16/23

* summary of relations amoung $\rho, V, \vec{E}$

$\boxed{\rightarrow \text{dipoles and more!}}$

dipole:



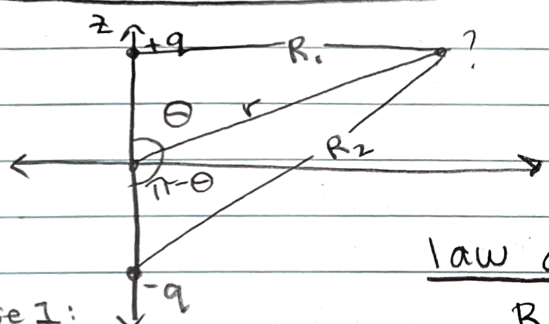
(for far-field): circles should be perpendicular w/field lines

spacing should get wider as you get further out

- net charge = zero
  - ↳ no lines going to ∞
    - ↳ no lines directly through the point charges (symmetry)

if there were a line directly through tilt the lines to correct

$\boxed{\text{electric potential}}$



$$V = \left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{q}{R_1} - \frac{q}{R_2}\right) = \left(\frac{q}{4\pi\epsilon_0}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

WE WANT V IN TERMS OF (r, θ), WE NEED GEOMETRY:

law of cosines:

$$R_1^2 = r^2 + \left(\frac{d}{2}\right)^2 - 2r\frac{d}{2}\cos\theta$$

$$\boxed{R_1 = \sqrt{r^2 + \left(\frac{d}{2}\right)^2 - rd\cos\theta}}$$

$$R_2^2 = r^2 + \left(\frac{d}{2}\right)^2 - 2r\frac{d}{2}\cos(\pi - \theta)$$

$$\boxed{R_2 = \sqrt{r^2 + \left(\frac{d}{2}\right)^2 + rd\cos\theta}}$$

Case 1: far away ("r" much larger than "d")

→ drop all $\left(\frac{d}{r}\right)^2$ terms:

$$R_1 = r\sqrt{1 + \left(\frac{d}{2r}\right)^2 - \frac{d}{r}\cos\theta}$$

$$\simeq r\sqrt{1 - \frac{d}{r}\cos\theta}$$

$$R_2 \simeq r\sqrt{r + \frac{d}{r}\cos\theta}$$

→ approximating the potential:

$$V \simeq \left(\frac{q}{4\pi\epsilon_0}\right)\left(\frac{1}{r}\right)\left(\frac{1}{\sqrt{1 - \frac{d}{r}\cos\theta}} - \frac{1}{\sqrt{1 + \frac{d}{r}\cos\theta}}\right)$$

$$\simeq \left(\frac{q}{r4\pi\epsilon_0}\right)\left(\left[\cancel{1} + \left(-\frac{1}{2}\right)\left(-\frac{d}{r}\cos\theta\right)\right] - \left[\cancel{1} + \left(-\frac{1}{2}\right)\left(\frac{d}{r}\cos\theta\right)\right]\right)$$

$$\simeq \boxed{\left(\frac{qd}{4\pi\epsilon_0}\right)\left(\frac{\cos\theta}{r^2}\right)}$$

→ the dipole moment

Taylor series expansion:

$$(1+x)^n \simeq 1 + nx$$

$$x = \frac{d}{r}\cos\theta$$

$$n = -1/2$$

le gendre poly: P_1

Ideal dipole: $\boxed{V_{dipole} = \left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{P\cos\theta}{r^2}\right)}$

p: the dipole moment

$p = |\vec{p}| = qd$

should result in a circle (look back at e-field diagram) OR look at book for accurate pics

$\boxed{\text{finding the } \vec{E}\text{-field}}$ $\qquad \vec{E} = -\nabla V$

here: $\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{\partial}{r\partial\theta} + \cancel{\ldots}$

$$-\nabla V = \left(\frac{P}{4\pi\epsilon_0}\right)\left(\left(\frac{2\cos\theta\hat{r}}{r^3}\right) + \left(\frac{\sin\theta\hat{\theta}}{r^3}\right)\right)$$

$$= \left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{P}{r^3}\right)\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$$

→ Still far-field: (not dropping as many terms: do better!)

$$\frac{1}{R_1} - \frac{1}{R_2} = \frac{1}{r}\left(\frac{1}{\sqrt{1+\left(\frac{d}{2r}\right)^2 - \frac{d}{r}\cos\theta}} - \frac{1}{\sqrt{1+\left(\frac{d}{2r}\right)^2 + \frac{d}{r}\cos\theta}}\right)$$

using COOL MATH FACT: $\boxed{\text{MULTI-POLE EXPANSION}}$

Le Gendre Polynomials

$$(1+h^2-2hx)^{-1/2} = P_0(x) + hP_1(x) + h^2P_2(x) + \ldots$$

$$= \sum_{l=0}^{\infty} h^l P_l(x)$$ generating function

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{3}{2}(x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3-3x)$

(more info in textbook)

→ In example:

(for R_1): $h = \frac{d}{2r}$ and $x = \cos\theta$

(for R_2): $h = \frac{-d}{2r}$ and $x = \cos\theta$

$$\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \left(\frac{1}{r}\right)\left[\left(\sum_{l=0}^{\infty} h^l P_l\cos\theta\right) - \left(\sum_{l=0}^{\infty}(-h)^l P_l\cos\theta\right)\right]$$

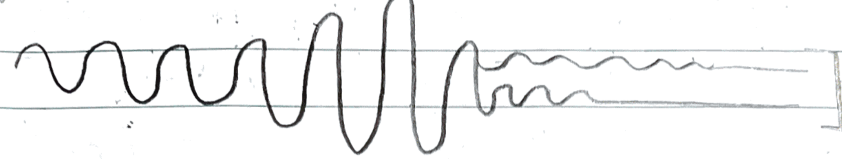
for odd: $= \left(\frac{2}{r}\right)\sum_{l=1}^{\infty} h^l P_l(\cos\theta)$

$$V = \left(\frac{q}{4\pi\epsilon_0}\right)\left(\frac{2}{r}\right)\left(\left(\frac{d}{2r}\right)\cos\theta + \left(\frac{d}{2r}\right)^3 P_3(\cos\theta) + \ldots\right)$$

$$= \left(\frac{1}{4\pi\epsilon_0}\right)\left(\underbrace{\frac{P\cos\theta}{r^2}}_{\text{dipole moment}} + \underbrace{\frac{qd^3}{4r^3}P_3(\cos\theta)}_{\text{octapole moment}} + \ldots\right)$$

fun application

gravitational wave forms: black hole memory



should be a displacement of strain

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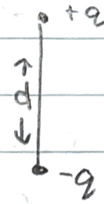
last time: dipoles 2/8/23

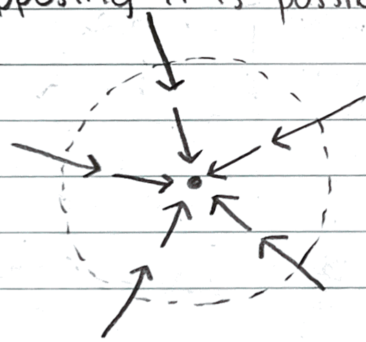
multiple expansion:

$\left(\frac{d}{r}\right)^{\ell} \underbrace{P_\ell(\cos\theta)}_{\text{legendre polynomial}}$

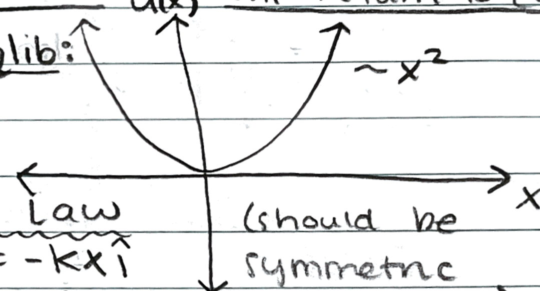


today: $\boxed{\text{theorem}}$ IS IT IMPOSSIBLE TO CONSTRUCT AN ELECTROSTATIC FIELD IN EMPTY SPACE THAT WILL HOLD A CHARGED PARTICLE IN STABLE EQUILIBRIUM

supposing it is possible:



stable eqlib:



U(x) will return to eql

Hooke's Law

$F = -kx\hat{i}$

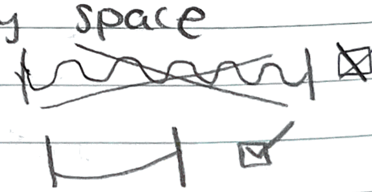
(should be symmetric parabola)

by GAUSS' LAW: $\oint \vec{E}\cdot d\vec{a} = \frac{q_{encl}}{\epsilon_0} \neq 0 =$ negative (all field lines in)

$\Rightarrow q_{encl} \neq 0$

$\Rightarrow$ contradicts that field is in empty space

AND V has no local minima



Conductors + insulators

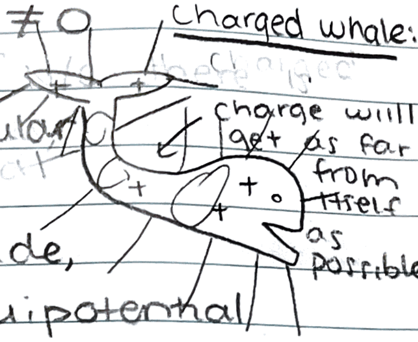
2/8/23

$\boxed{\text{conductors}}$: eg. metal, tap water, humans (mostly water)

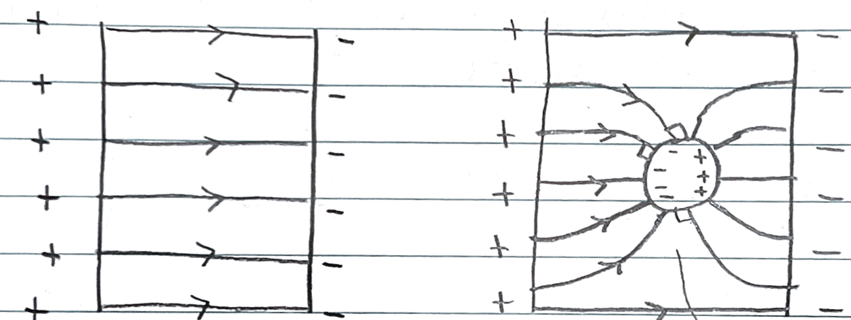
STATICS

* waiting till charge has stopped moving *

- on conductors, charges are free to move
- charge inside: $\rho = 0$
  ↳ charge runs to surface $\sigma \neq 0$
- $E_{inside} = 0$ (averaged over atoms)
- on the boundry, $\vec{E}$ is perpendicular / normal
- the potential is constant inside, and the boundry is equipotential



example:



before (no interference)

after (w/ $\vec{E} = 0$), circular surface added

*field lines must be perpendicular to surface

FINDING E and V from ρ:

$$\Delta V = -\int \vec{E}_1 \cdot d\vec{s}$$

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{encl}}{\epsilon_0} \quad \text{OR} \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{E} = -\vec{\nabla} V \implies \vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot (-\vec{\nabla} V) = -\nabla^2 V = \frac{\rho}{\epsilon_0}$$

often, we solve for empty space, so

$\rho \to 0$: $\boxed{\nabla^2 V = 0}$ LAPLACE'S EQUATION

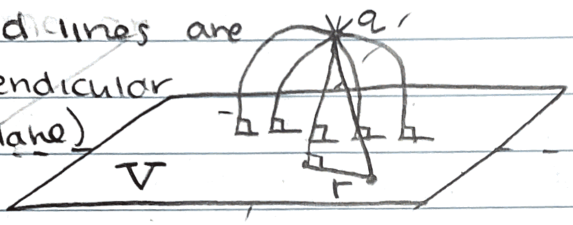
2/8/23

$\boxed{\text{EARNSHAW THEOREM}}$ The V is uniquely determined by $\nabla^2 V = 0$ in a region R, if V (or its normal derivative $\vec{\nabla} \cdot V(\hat{n})$) is set on ∂R (boundary of region R)

example: Image charge

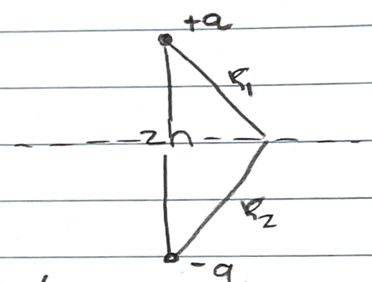
→ what is the $\vec{E}$-field when a charge "q" is placed "h" above a conducting plane?

(field lines are perpendicular to plane)



→ looks like half a dipole moment!

dipole:



- Check that boundary conditions are satisfied:

$$V = \left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{q}{R_1} - \frac{q}{R_2}\right) = \left(\frac{q}{4\pi\epsilon_0}\right)\left(\frac{1}{\sqrt{h^2+r^2}} - \frac{1}{\sqrt{(-h)^2+r^2}}\right)$$