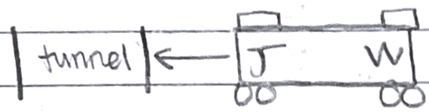
3/6/23

so far we have looked @:

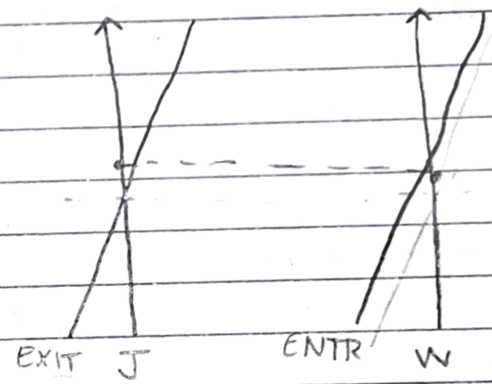
- slip in simultaneity
- time dilation
- length contraction
- addition of velocities (red-shift)

today: lorentz transformation: $(t, x) \longleftrightarrow (t', x')$

→ train example from last time: train in tunnel (3/3/23)

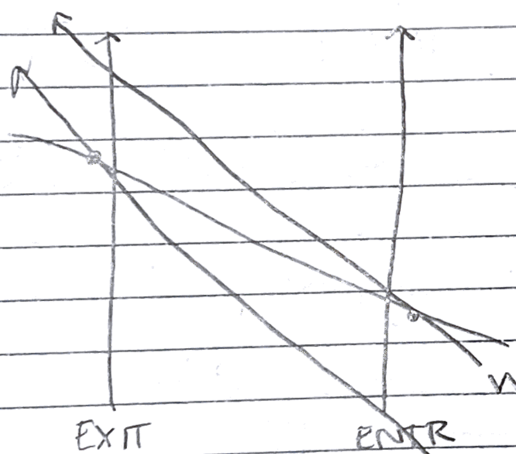


train frame:



(should all be straight parallel lines)

tunnel frame:



→ train contracts

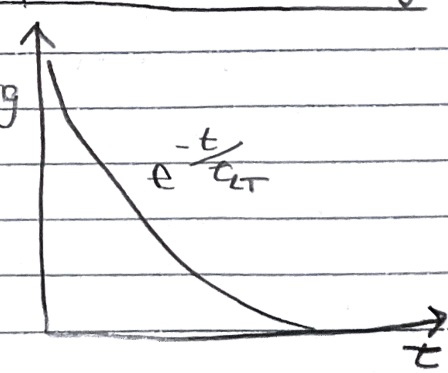
example 2: Muons

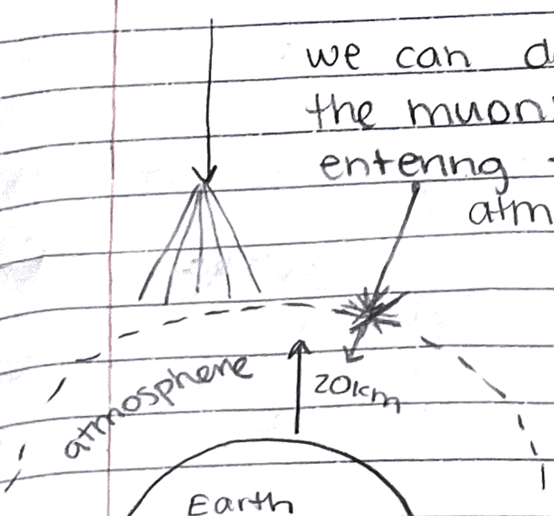
lives; $t_{LT} \simeq 2.2 \mu s$

$\mu \longrightarrow e^- + \nu + \bar{\nu}$

↑ electron

lifetime = exponential decay:





→ they don't all die simultaneously

(cont. on back)

velocity of the muon: $V \simeq 0.99c$

$$t = \frac{20{,}000\,m}{(0.99)(3\times10^8\,m/s)} = 67\mu s$$

$$\frac{t}{t_{LT}} = \frac{67\mu s}{2.2\mu s} \simeq 30 \rightarrow \text{fraction } f = e^{-t/t_{LT}} = e^{-30} \simeq 10^{-13}$$

= highly unlikely to observe muons, however we do!

→ we forgot time dilation: $\gamma = \frac{1}{\sqrt{1-(0.99)^2}} \simeq 7.1$

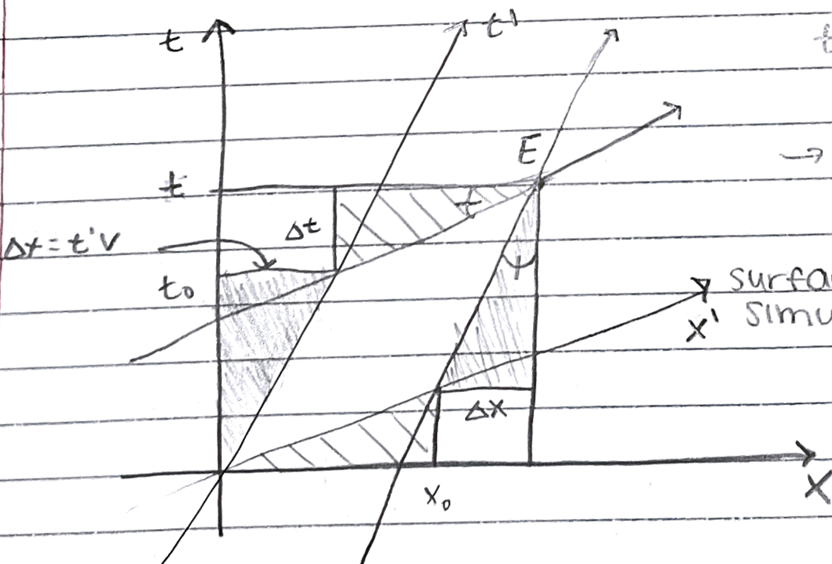
$$\frac{t}{t_{LT}} = \frac{67\mu s}{(\gamma t_{LT})}$$ — lifetime in earth's frame

$$= \frac{67\mu s}{(7.1)(2.2)} \simeq 4.2 \Rightarrow f = e^{-4.2} \simeq 0.015$$

now explains why a few make it to the surface

fun fact: altitude can be measured by muon rate

deriving lorentz transformations:



$t = t_0 + \Delta t$

$x = x_0 + \Delta x$

→ shaded triangles are congruent (2 sets of congruent triangles)

→ surface of simultaneity

$$\Delta t = \frac{V x_0}{c^2}$$

$t_0 = \gamma t'$ (time dilation)

$x_0 = \gamma x'$ (length contraction)

$\Delta x = V t_0$

finding relations:

$$t = \gamma t' + \frac{V x_0}{c^2}$$
$$= \gamma t' + V/c^2\,\gamma x'$$
$$= \boxed{\gamma(t' + V/c^2\,x')}$$

$$x = \gamma x' + V t_0$$
$$= \gamma x' + V\gamma t'$$
$$= \boxed{\gamma(x' + V t')}$$

$$\boxed{\begin{array}{l} y = y' \\ z = z' \end{array}}$$

last time: lorentz transformations 3/8/23

$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$ both mix

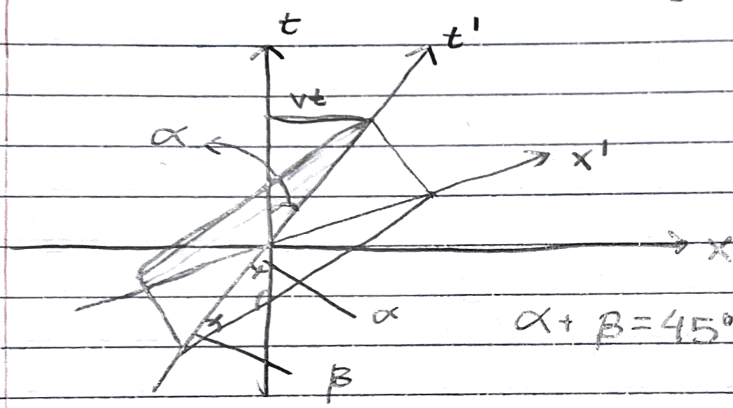
$x = \gamma(x' + vt')$ space & time

$y = y'$ $\Rightarrow$ spacetime is more physical

$z = z'$

today: energy + momentum - invariant variable

- tilt of simultaneity



$\alpha + \beta = 45°$

$\Rightarrow$ Is there something that is the same in all frames?

yes: $\boxed{\text{invariant variable} \quad \Delta s^2}$

$$\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = \Delta s'^2$$

proof that $\Delta s^2 = \Delta s'^2$

can drop b/c we know are the same

$$\Delta s^2 = -c^2\gamma^2\left(\Delta t' + \frac{v\Delta x}{c^2}\right)^2 + \gamma^2(\Delta x' + v\Delta t')^2 + \Delta y'^2 + \Delta z'^2$$

$$= \frac{-c^2}{1-\frac{v^2}{c^2}}\left(\Delta t'^2 + \cancel{2v\frac{\Delta t'\Delta x'}{c^2}} + \frac{v^2}{c^2}\Delta x'^2\right) + \frac{1}{1-\frac{v^2}{c^2}}\left(\Delta x'^2 + \cancel{2\Delta x\Delta t v} + v^2\Delta t'^2\right)$$

$$= \frac{\Delta t'^2}{1-\frac{v^2}{c^2}}\left(-c^2+v^2\right) + \frac{\Delta x'^2}{1-\frac{v^2}{c^2}}\left(-\frac{v^2}{c^2}+1\right)$$

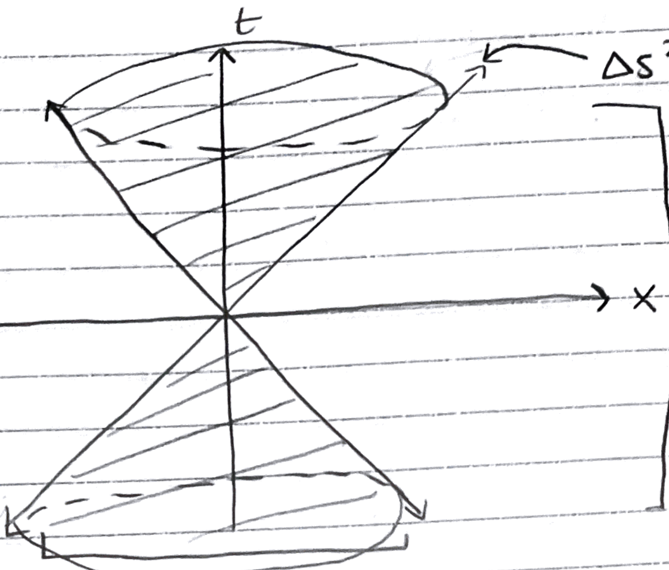
$$= -c^2\Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2 = \Delta s'^2$$

This implies Δs^2 is invariant as we desired

(graph of invariant on next page)

(on lines) 3/8/23

$\Delta s^2 = 0$: light-like or "Null"



(unshaded regions)
$\Delta s^2 > 0$

$\Delta s^2 < 0$: time like
(diagonal shaded regions)

energy-mass momentum:

- non-relativityistic (newtonian):
  - → mass = conserved
  - → momentum = conserved
  - → energy = conserved (elastic collisions)

ex: $\vec{p} = \underbrace{m_1 \vec{V}_0 + m_2 \vec{V}_1}_{\text{before}} = \underbrace{m_1 \vec{V}_2 + m_2 \vec{V}_3}_{\text{after}}$

- In a diff. frame:

$$\vec{V}' = \vec{V} + \vec{u}$$

$$p' = m_1(\vec{V}_{10} + u) + m_2(\vec{V}_{20} + \vec{u}) = p + M\vec{u}$$

⇒ In special relativity:

$$\vec{V}' = \frac{V+u}{1 - \frac{Vu}{c^2}}$$

this does not work! (something similar does)

idea: $V = \frac{dx}{dt}$ — non-relativistic does transform this — does not transform this

(continued on next page)

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goals: we want new quantities, such that:

1) we recover non-relativistic m, KE, $\vec{p}$

2) conservation is valid in all frames

try: $\vec{p} = \gamma m \vec{v}$ ← generalized momentum

$p^0 = \gamma m$

suppose these are conserved in one frame.

Therefore: $\vec{p}' = \gamma' m \vec{v}' = \gamma(\vec{p} + V p_0)$ [using Lorentz]

$p_0' = \gamma' m = \gamma\left(p_0 + \frac{V\vec{p}}{c^2}\right)$

lots of messy algebra needed to get between these two

→ do these new momenta reduce to the non-relativistic ones? (when $V/c \ll 1$)

$$\vec{p} = \gamma m \vec{v} \xrightarrow[\substack{V = \text{small} \\ \gamma = 1}]{} m\vec{v}$$

and:

$$\gamma = \frac{1}{\sqrt{1 - V^2/c^2}} = \left(1 - \frac{V^2}{c^2}\right)^{-1/2} \simeq 1 + \frac{1}{2}\left(\frac{V^2}{c^2}\right)$$

$$c^2 p^0 = c^2 \gamma m$$
$$= c^2\left(1 + \frac{1}{2}\frac{V^2}{c^2}\right)m$$
$$= mc^2 + \tfrac{1}{2}mv^2$$

★ so: $k = \frac{1}{2}mv^2$ is just leading term of KE

mc^2 = energy!

mass is NOT conserved: mass ⟷ KE

"mass" or "rest-mass" (mass of obj wrt to viewer at rest) last time: energy-momentum 3/10/23

$$\left\{\begin{array}{l} E = \gamma m c^2 \ (= c^2 p^0) \\ \vec{p} = \gamma m \vec{v} \end{array}\right\} \rightarrow \text{reduce to non-relativistic } E \text{ and } P \text{ when } v/c \ll 1$$

→ what about a particle w/no mass?

= replaced by $E = |p|c$ (remember $E = h\nu = \hbar\omega$ for photons)

★NOTE: $\Delta s^2 = \Delta s'^2 = -\Delta t^2 + \Delta x^2$ = invariant

↳ same in all frames!

try: $-E^2 + p^2 = -\left(\frac{1}{1-v^2}\right) m^2 v^2 = m^2\left(\frac{-1+v^2}{1-v^2}\right) = -m^2$

↑ this mass is an invariant

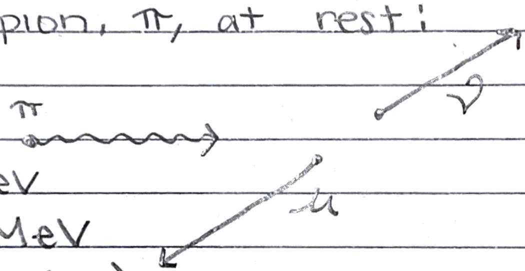
$$E^2 - p^2 = m^2$$

lorentz transformations (in E-p space):

$$p^{0\prime} = \gamma\left(p^0 + \frac{vp}{c^2}\right)$$

$$p' = \gamma(p + vp^0)$$

example: pion, π, at rest: → will decay into a muon, μ, and a neutrino, ν



$m_\pi \simeq 140\,\text{MeV}$

$m_\mu \simeq 106\,\text{MeV}$

$m_\nu = 0$ (massless)

⟹ what is the speed of the muon? $\boxed{v_\mu = \frac{|P_\mu|}{E_\mu}}$

decay: $\pi \rightarrow \mu + \nu$

Conservation of:

$$E: \ E_\pi = E_\mu + E_\nu$$

starts at rest

$$\vec{P}: \ \vec{P}_\pi = \vec{P}_\mu + \vec{P}_\nu, \text{ but } \vec{P}_\pi = 0 \Rightarrow |\vec{P}_\mu| = |\vec{P}_\nu|$$

$E_\pi = m_\pi$

$E_\mu = \sqrt{P_\mu^2 + m_\mu^2}$

$E_\nu = |P_\nu| = |P_\mu|$

by conservation of energy:

$$m_\pi = \sqrt{P_\mu^2 + m_\mu^2} + |\vec{P}_\nu|$$

$$(m_\pi - P_\mu)^2 = P_\mu^2 + m_\mu^2$$

$$m_\pi^2 - 2P_\mu m_\pi + \cancel{P_\mu^2} = \cancel{P_\mu^2} + m_\mu^2$$

$$\Rightarrow \boxed{P_\mu = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}}$$

cont. on back

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while: $E_\mu^2 = P_\mu^2 + m_\mu^2 = \left(\frac{m_\pi^2 - m_\mu^2}{2m_\pi}\right)^2 + m_\mu^2$

$$= \frac{m_\pi^4 - 2m_\pi^2 m_\mu^2 + m_\mu^4 + 4m_\pi^2 m_\mu^2}{4m_\pi^2}$$

$$= \frac{(m_\pi^2 + m_\mu^2)^2}{(2m_\pi)^2}$$ Therefore, $E_\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi}$

$$V = \frac{|\vec{P}_\mu|}{E_\mu} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2} \approx 0.27$$

$\boxed{V = 0.27c}$ "relative particle kinematics"