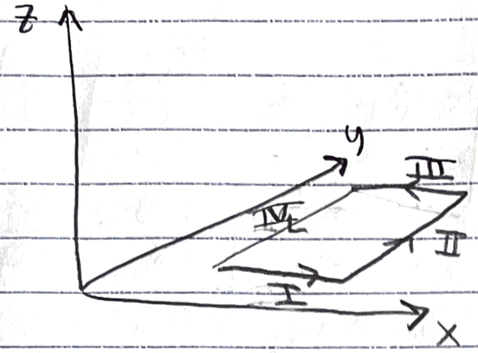
denvation of local circulation continued 4/10/23



$\oint \vec{V}\cdot d\vec{s}$ $\quad d\vec{s} = \hat{i}\,dx$

$\Gamma = \int_{loop} \vec{v}\cdot d\vec{s} =$

$\int_{I} \vec{v}\cdot d\vec{s} + \int_{II} \vec{v}\cdot d\vec{s} + \int_{III} v\cdot d\vec{s} + \int_{IV} v\cdot d\vec{s}$

on I: $\int_{I} \vec{v}\cdot d\vec{s} \simeq \int_{0}^{\Delta x} (V_x \frac{\partial V_x}{\partial x}(\Delta x)\,dx$

$= V_x(\Delta x + \frac{\partial V_x}{\partial x})\Delta x^2$

on II:

$\int_{II} \vec{v}\cdot d\vec{s} \simeq \int_{II} V_y\,dy \simeq \int_{II} V_y + \frac{\partial y}{\partial x}\Delta x + \frac{\partial V_y}{\partial y}(\Delta y)\,dy$

$= V_y\Delta y + \frac{\partial V_y}{\partial y}\Delta y^2 + \frac{\partial y}{\partial x}\Delta x\Delta y$

on III: $\int_{III} \vec{V}\cdot d\vec{s} = -\int_{III} V_x\,dx \simeq -\int_{III} (V_{x(x,y)} \frac{\partial V_x}{\partial x}\Big|\Delta x + \frac{\partial V_x}{\partial y}\Big|\Delta y)\,dx$

$= -V_x\Delta x - \frac{\partial V_x}{\partial x}\Delta x^2 - \frac{\partial V_x}{\partial y}\Delta x\Delta y$

on IV: $\int_{IV} \vec{v}\cdot d\vec{s} = -\int_{IV} V_y\,dy \simeq -\int_{IV} (V_y + \frac{\partial V_y}{\partial y}\Big|\Delta y)\,dy$

$= -V_y\Delta y - \frac{\partial V_y}{\partial y}\Delta y^2$

around entire loop:

$\Gamma = \int_{100} \vec{v}\cdot d\vec{s} = \cancel{V_x(\Delta x + \frac{\partial V_x}{\partial x})\Delta x^2} + \cancel{V_y\Delta y} + \cancel{\frac{\partial V_y}{\partial y}\Delta y^2} + \frac{\partial V_y}{\partial x}\Delta x\Delta y$
$- \cancel{V_x\Delta x} - \cancel{\frac{\partial V_x}{\partial x}\Delta x^2} - \frac{\partial V_x}{\partial y}\Delta x\Delta y - \cancel{V_y\Delta x} - \cancel{\frac{\partial V_y}{\partial y}\Delta y^2}$

* $\hat{n} = \hat{a} = \frac{\vec{a}}{|\vec{a}|}$

$= \frac{\partial V_y}{\partial x}\Delta x\Delta y - \frac{\partial V_x}{\partial y}\Delta x\Delta y$

$= \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\Delta x\Delta y$

$$\boxed{\vec{\nabla}\times\vec{V}\cdot\hat{u} = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = (\vec{\nabla}\times\vec{V})_z}$$

$$\boxed{\nabla^2\vec{A} = -\mu_0\vec{J}}$$

cartesian coordinates

in cartesian co-ordinates:

$\vec{\nabla} \times \vec{\nabla} =$ (cross-product)

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \hat{i}\left|\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right| - \hat{j}\left|\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right| + \hat{k}\left|\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right|$$

example: $\nabla \times \vec{A}$

$$A = (xz^2 - x^2 z)\hat{j} + (2xyz - \tfrac{1}{2}xy^2)\hat{k}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & xz^2 - x^2 z & 2xyz - \frac{1}{2}xy^2 \end{vmatrix} = \hat{i}\left((2xz - xy) - (2xz - x^2)\right) - \hat{j}\left(\left(2yz + \frac{y^2}{2}\right) - 0\right) + \hat{k}\left((2xz - z^2) - 0\right)$$

$$\boxed{= (-xy + x^2)\hat{i} - \left(2yz + \frac{y^2}{2}\right)\hat{j} + (2xz - z^2)\hat{k} \neq 0}$$

(examples of vector field drawings in book!)

for statics:

Guass': $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ $\quad \vec{\nabla} \times \vec{E} = 0$

if $\vec{E} = -\vec{\nabla} V$ for electric potential V:

$$-\vec{\nabla} \cdot \vec{\nabla} V = \frac{\rho}{\epsilon_0} \quad \text{or} \quad \nabla^2 V = \frac{-\rho}{\epsilon_0}$$

OR $\nabla^2 V = 0$ + BC's

for a point charge: $V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$

$\nabla \times \vec{B} = \mu_0 \vec{J}$ $\quad$ Is there an analog of V?

$\vec{\nabla} \cdot \vec{B} = 0$ $\quad$ yes! There is a vector potential $\vec{A}$:

$$\vec{B} = \nabla \times \vec{A}$$

identity: $\nabla \times (\nabla \times \vec{A}) = -\nabla^2 \vec{A} + \vec{\nabla}(\nabla \vec{A})$ AND $\vec{\nabla} \times (\vec{\nabla}\lambda) = 0$

$$\boxed{\Rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\lambda \quad \text{GAUGE INVARIANCE}}$$

last-time: curl $\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$ 4/12/23

today: BIOT-SAVART

(local ampere's law) $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$

$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$

$\vec{\nabla}\cdot\vec{B} = 0$ (no magnetic monopoles) (transfer btwn there using stoke's theorem)

☆ ALL ∇ should have arrows on top → they are vector units ☆

argument:

define: vector potential, $\vec{A}$: $\vec{B} = \vec{\nabla}\times\vec{A}$

$$\underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{A})}_{-\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})} = \mu_0\vec{J}$$

$$-\nabla^2\vec{A} + \vec{\nabla}(\underbrace{\cancel{\vec{\nabla}\cdot\vec{A}}}_{0}) = \mu_0\vec{J}$$

choose GAUGE of $\vec{A}$, such that $\nabla\cdot\vec{A} = 0$

$$\Rightarrow \underbrace{\vec{\nabla}^2\vec{A} = -\mu_0\vec{J}}_{\text{magnetic potential}} \leftrightarrow \underbrace{\nabla^2 V = \frac{\rho}{\epsilon_0}}_{\text{electric potential}}$$

$\nabla^2 V = 0$ + boundry conditions

therefore:

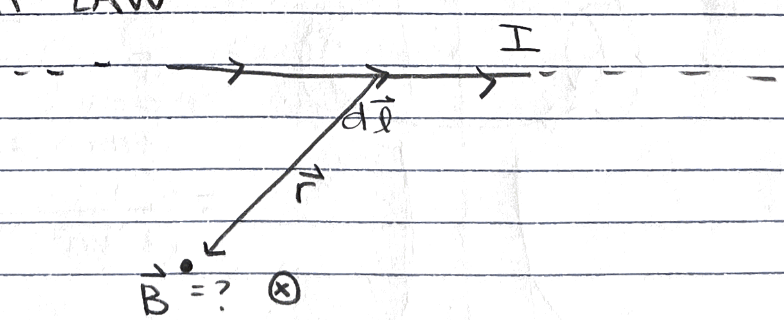
$$\vec{A} = \frac{-\mu_0}{4\pi}\int \frac{\vec{J}\,dV}{r} \qquad V = \frac{1}{4\pi\epsilon_0}\int\frac{\rho\,dV}{r}$$

w/some additional thought + direction

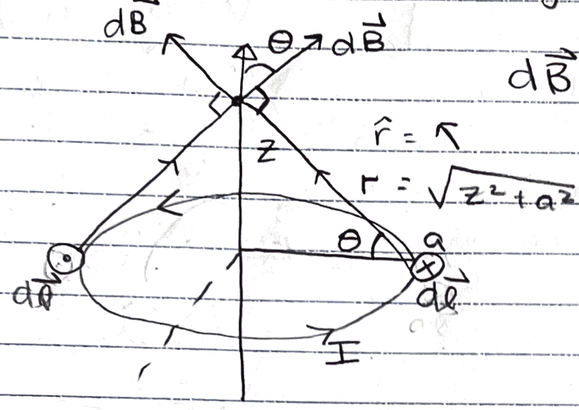
$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{l}\times\hat{r}}{r^2}}$$

BIOT-SAVART LAW

example:



example: find $\vec{B}$-field on axis of a current carrying loop 4/12/23



$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\,d\vec{\ell} \times \hat{r}}{r^2}$$

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I\,d\vec{\ell} \times \hat{r}}{r^2}$$

$$= \frac{\mu_0}{4\pi} \int \frac{I\,d\ell\,r\,\cos\theta}{r^2}$$

$I\,d\ell \times \hat{r} = I\,d\ell\,\cos\theta\,\hat{k}$ (all of the horizontal components cancel out when integrating around the loop, leaving only vertical)

$$= \frac{\mu_0}{4\pi} \int \frac{I\,d\ell\,\cos\theta\,\hat{k}}{r^2}$$

$$= \frac{I\mu_0 \cos\theta}{4\pi r^2} \int d\ell\,\hat{k}$$

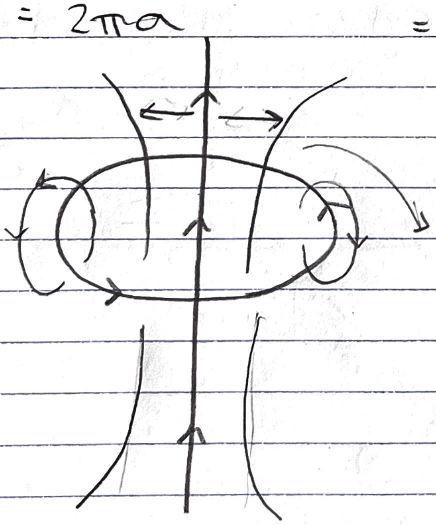
$$= \frac{I\mu_0}{4\pi r^2}\left(\frac{a}{r}\right) \int d\ell\,\hat{k}$$

$$= \frac{\mu_0 I a}{4\pi(z^2+a^2)^{3/2}} \int d\ell\,\hat{k}$$

$\int d\ell = 2\pi a$

$$= \frac{\mu_0 I a}{4\pi(z^2+a^2)^{3/2}} (2\pi a)\,\hat{k}$$

$$= \boxed{\frac{\mu_0 I a^2}{2(z^2+a^2)^{3/2}}\,\hat{k}}$$



spacing of field lines must following:
(from z-axis to next field line):

$$= \frac{1}{(z^2+a^2)^{3/2}}$$

summarizing what we've done so far: 4/12/23

$\boxed{\text{electrostatics}}$:

Coulomb's law:

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{r} \longrightarrow$$

$\boxed{\text{e-fields}}$:

Gauss' law:

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}}$$

divergence theorem

$$\oint \vec{E}\cdot d\vec{a} = \frac{q_{encl}}{\epsilon_0}$$

over volume

$$\int_V \vec{\nabla}\cdot\vec{E}\,dV = \oint_{\partial V} \vec{E}\cdot d\vec{a}$$

$$q_{encl} = \int_V \rho\, dV$$

$$\boxed{\nabla\times E = 0}$$

$\boxed{\text{electric-potentials}}$

"V"

$\boxed{\text{special relativity}}$:

* "moving objects shrink"
* "moving clocks slow"

↳ e-field transformations:

$$E'_\perp = \gamma E_\perp$$

but, $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) \rightarrow$ must be transformed (from switching frames) magnetic fields

$\boxed{\text{magnetics}}$:

$\vec{B}$ comes from moving charges

→ quantitatively: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$

local form of Ampere's law: $\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$

BIOT-SAVART: $\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\, d\vec{l}\times\hat{r}}{r^2}$ $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$

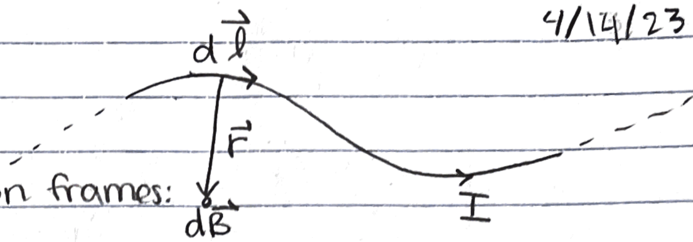
$\boxed{\text{dynamics}}$

- what changes in these 4-equations?
- what are the transformation laws for $\vec{E}$ and $\vec{B}$?

4/14/23

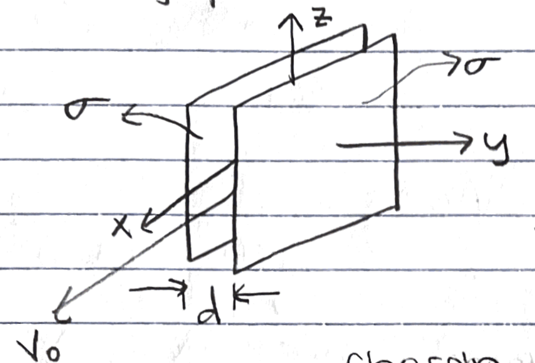
<u>last time</u>: Biot - savart

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \hat{r}}{r^2}$$



<u>today</u>: transformations btwn frames: $\vec{E}$ & $\vec{B}$

→ sliding plates: $\sigma > 0$

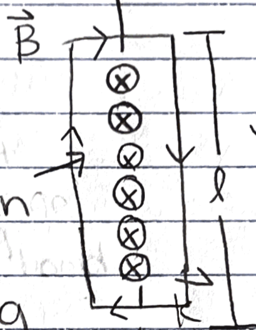


all in frame "S"

$\vec{K} = \sigma \vec{V}_0$

→ moving in x-direction at speed V_0

$$\boxed{\vec{E} = \frac{\sigma}{\epsilon_0}\hat{j} \quad \text{OR} \quad E_y = \frac{\sigma}{\epsilon_0}}$$



chosen amperian loop, enclosing the charge

(for single current sheet)

$$\underbrace{\oint \vec{B} \cdot d\vec{l}}_{} = \mu_0 I_{encl}$$

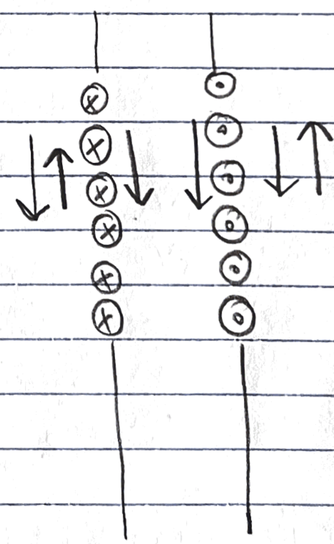
$$2B\cancel{l} = \mu_0 K \cancel{l}$$

$$= |\vec{B}| = \frac{\mu_0 K}{2}$$

(for two current sheets):

★relies on capacitors running off to ∞★

$$\boxed{\vec{B} = \mu_0 K \cdot \hat{k} \quad \text{OR} \quad B_z = \mu_0 K}$$



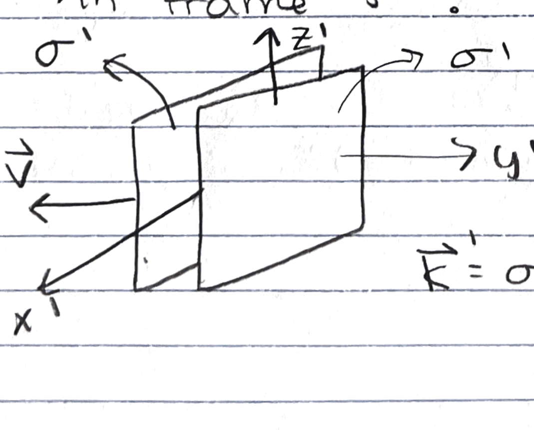
⇒suppose instead of σ, you have ΔV and d:

$$\boxed{|E| = \frac{\Delta V}{d}}$$

★ $\vec{E} = -\nabla V$

$V = -\int \vec{E} \cdot d\vec{l}$

⇒in frame "S'": $\vec{V} = V\hat{\imath}$



(continued on back!)

$$V_0' = \frac{V - V_0}{1 - VV_0}$$

$\vec{K}' = \sigma' \vec{V}_0'$

☆ $\frac{1}{\mu_0 \epsilon_0} = c^2$ ☆

gamma factors: 4/14/23

$\sigma' = \gamma_0' \frac{\sigma}{\gamma_0}$

$\gamma_0 = \frac{1}{\sqrt{1-v_0^2}}$ $\gamma_0' = \frac{1}{\sqrt{1-v_0'^2}}$

$\sigma' = \cancel{\gamma_0}\gamma(1-vv_0)\frac{\sigma}{\cancel{\gamma_0}}$

$= \boxed{\gamma\sigma(1-vv_0)}$

$\gamma_0' = \frac{1}{\sqrt{1-\frac{(v_0-v)^2}{(1-vv_0)^2}}}$

$= \frac{1-vv_0}{\sqrt{1-2vv_0+v^2v_0^2-v_0^2-v^2+2vv_0}}$

$= \frac{1-vv_0}{\sqrt{(1-v_0^2)(1-v)^2}} = \boxed{\gamma_0\gamma(1-vv_0)}$

$E_y' = \frac{\sigma'}{\epsilon_0} = \frac{\gamma\sigma(1-vv_0)}{\epsilon_0}$

$B_z' = \mu_0 K' = \mu_0\gamma\sigma(1-vv_0)\left(\frac{v_0-v}{1-vv_0}\right)$

$B_z' = \mu_0\gamma\sigma(v_0-v) = \gamma(\mu_0\sigma v_0 - \mu_0\sigma v)$

$E_y' = \gamma\left(\frac{\sigma}{\epsilon_0} - \frac{vv_0\sigma}{\epsilon_0}\right) = \gamma\left(E_y - \frac{vB_z}{\mu_0\epsilon_0}\right)$

$B_z' = \gamma(B_z - vE_y\mu_0\epsilon_0)$

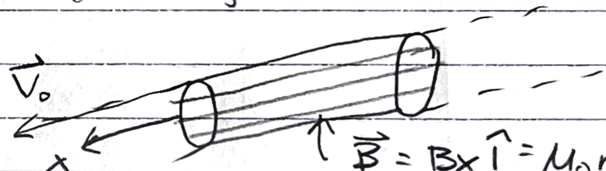
What happens to B_x'? (looking at solenoid)

$$\boxed{\begin{aligned} E_x' &= E_x \\ E_y' &= \gamma(E_y - vB_z) \\ E_z' &= \gamma(E_z + vB_y) \\ B_x' &= B_x \\ B_y' &= \gamma\left(B_y + \frac{vE_z}{c^2}\right) \\ B_z' &= \gamma\left(B_z - \frac{vE_y}{c^2}\right) \end{aligned}}$$

LORENTZ TRANSFORMATIONS OF $\vec{E}$ & $\vec{B}$

(can swap $\vec{E} \leftrightarrow c\vec{B}$)

☆ONLY IF $\vec{v} = v\hat{i}$ (x-direction)☆



$\vec{B} = B_x\hat{i} = \mu_0 n I \hat{i}$

turns per unit length

= both length contraction & time dilation ⇒ cancel each other out in B' calculation

$B_x' \to \mu_0 n' I' \hat{i}'$

$= \mu_0 \cancel{\gamma} n \frac{I}{\cancel{\gamma}} \hat{i} = B_x$