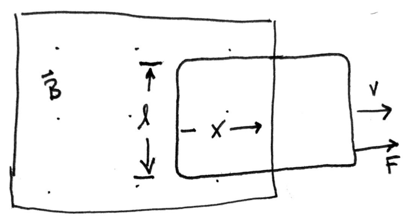
Solutions:

(1) Done in book. However, your solutions need to be clear and complete.

(2) This is a scaling problem. First, the physics: As the frame moves out of the field the flux through the loop decreases. This induces an EMF that drives a current in the loop. This current then enjoys a $I\ell \times B$ force that resists the motion. (Check this by choosing a direction for $\mathbf{B}$, say up. Then, I runs down and $I\ell \times B$ points to the left.) Here's a sketch,



Now working through this to obtain the scaling,

$$\mathcal{E} = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{a} = -\frac{d}{dt}Bx\ell = -Bv\ell$$

While, $\mathcal{E} = IR$ so the current is

$$I = \frac{Bv\ell}{R}.$$

The left side of the loop has the force $I\ell B$, since the vectors are perpendicular, and the applied force balances this magnetic one, $F = I\ell B$. Finally at terminal velocity $v = x/t$. Thus,

$$F = \frac{B^2 x\ell^2}{tR}.$$

The first statement about a 1 N force taking 1 s says

$$1 = \frac{B^2 x\ell^2}{R} \text{ so a 2 N force must mean } 2 = \frac{B^2 x\ell^2}{(1/2)R}$$

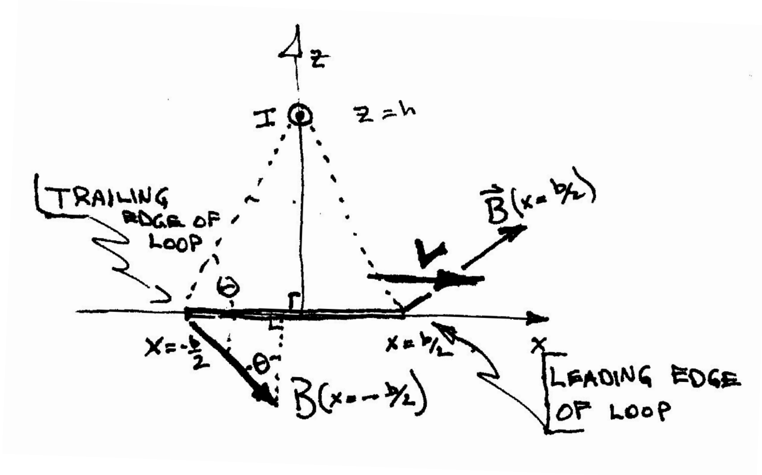
or $t = 1/2$ s. Since the resistivity is proportional to the resistance the brass rod loop would take 1/2 a Newton, since

$$\frac{1}{2} = \frac{B^2 x\ell^2}{2R_{Brass}} = \frac{B^2 x\ell^2}{2R_{Al}}.$$

Finally, changing the diameter changes the area and $R = \rho L/A$. Thus, when we double the diameter we increase the area by 4 and

$$\frac{B^2 x\ell^2 A}{\rho L} \to \frac{B^2 x\ell^2 4A}{\rho L} \implies F \to 4 \text{ N}.$$

(3) The current along the y axis (pointing out in this sketch) produces a B-field that circulates around the point $(x = 0, z = h)$ in the $x - z$ plane. Due to the symmetric arrangement, at the instant shown in the sketch, when the center of the square loop crosses the z axis, we have some hope to compute the flux, and its change.



The B-field is

$$B = \frac{\mu_o I}{2\pi r} \text{ or, in these coordinates } B = \frac{\mu_o I}{2\pi\sqrt{h^2 + (b/2)^2}}$$

in magnitude at the edges of the loop. Only the z components contribute to the flux. They are

$$B_z = \frac{\mu_o I}{2\pi\sqrt{h^2 + (b/2)^2}} \frac{b/2}{r} = \frac{\mu_o I b}{4\pi(h^2 + (b/2)^2)}.$$

At the trailing edge the B-field points down and to the right while at the leading edge it points up and to the right. If the loop moves the small distance vdt it looses a bit of downward flux on the trailing edge and gains a bit of upward flux on the leading edge. Both these increase the flux up through the loop. These have the same magnitude, thus

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -2\frac{b\, vdt\, B_z}{dt} = -\frac{\mu_o I b^2 v}{2\pi(h^2 + (b/2)^2)}.$$

The minus sign is Lenz's sign and tells us that the induced current will run CW as viewed from above, creating an induced B-field that points downward. (I know that the question asks for magnitude; I just wish it didn't.)

(4) More changing flux

(a) The area enclosed in the loop is bx, which increases as bdx/dt or bv. Since the field is uniform, that change in flux is Bbv, which is equal to (minus) the induced EMF. The current is $I = Bbv/R$. The $I\ell \times B$ force is simply $F = IBb = B^2b^2v/R$. This force opposes the motion. (Check this by choosing a direction for $\mathbf{B}$, say up. Then, I runs down and $I\ell \times B$ points to the left.) So,

$$F = \frac{dp}{dt} \implies m\frac{dv}{dt} = -\frac{B^2b^2}{R}v$$

This is a pretty easy differential equation to solve. It separates and is integrable. Integrating from $t = 0$ when $v = v_o$ to a time t and speed v,

$$\int_{v_o}^{v} \frac{dv'}{v'} = -\frac{B^2b^2}{mR}\int_0^t dt' \implies \ln\left(\frac{v}{v_i}\right) = -\frac{B^2b^2}{mR}t$$

So

$$v = v_o e^{-\alpha t} \text{ with } \alpha = \frac{B^2b^2}{mR}$$

It slows exponentially quickly.

(b) Let's integrate up to find the distance,

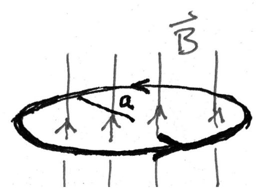
$$x = \int_0^\infty v(t)dt = \int_0^\infty v_o e^{-\alpha t} dt = -v_o/\alpha e^{-\alpha t}\big|_o^\infty = \frac{v_o m R}{B^2 b^2}.$$

(c) The initial kinetic energy is $1/2mv_o^2$ (if non-relativistic). This energy must end up as heat of the resistor. Let's check. The power dissipated is $P = I^2 R$ and the current is $I = Bbv/R = Bbv_o e^{-\alpha t}/R$ so the total energy is

$$\int_0^\infty I^2 R dt = \frac{B^2 b^2 v_o^2}{R} \int_0^\infty e^{-2\alpha t} dt = \frac{B^2 b^2 v_o^2}{R} \frac{mR}{2B^2 b^2} = \frac{1}{2} m v_o^2$$

Neat! It works!

(5) The ring with embedded charge q has a B-field threading through it, initially of magnitude B_o. I'll assume the ring has mass M. As this field is decreased, there is a circulatory induced electric field that applied a torque, rotating the ring.



First, to find the electric field we use Faraday's law

$$\oint_{\partial S} \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{a}.$$

The ring has radius a so the left hand side is

$$\oint_{\partial S} \mathbf{E} \cdot d\mathbf{s} = E \int ds = E 2\pi a.$$

Meanwhile the magnetic flux through the ring, and surface S, is,

$$\int_S \mathbf{B} \cdot d\mathbf{a} = B \int_S da = B\pi a^2.$$

Its time derivative is

$$\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{a} = \frac{dB}{dt} \pi a^2$$

The magnetic field is decreasing so the derivative $dB/dt < 0$. Using Faraday's law we have,

$$E = -\frac{dB}{dt} \frac{a}{2},$$

directed counterclockwise as shown due to Lenz's law - 'nature abhors a change in flux'. This electric field exerts forces on the charges embedded in the ring and thus a torque on the ring. The force per unit length is

$$\frac{F}{L} = \lambda E = -\frac{q}{2\pi a} \frac{dB}{dt} \frac{a}{2} = -\frac{q}{4\pi} \frac{dB}{dt}.$$

Hence the torque per unit length is

$$\frac{\tau}{L} = r \times \frac{F}{L} = -a \frac{q}{4\pi} \frac{dB}{dt}$$

This points upward along the axis of the ring (assuming q is positive). Integrating to find the total torque,

$$\tau = \int \frac{\tau}{L} ds = -a\frac{q}{4\pi}\frac{dB}{dt}2\pi a = -\frac{qa^2}{2}\frac{dB}{dt}.$$

The torque is equal to the change in angular momentum,

$$\tau = \frac{dL}{dt} \text{ with } L = I\omega$$

Using the hint we have that the torque is equal to

$$\tau = Ma^2\frac{d\omega}{dt} = -\frac{qa^2}{2}\frac{dB}{dt}.$$

Integrating both sides gives

$$M\int_0^{\omega_f} d\omega = -\frac{q}{2}\int_{B_o}^0 dB \implies \omega = \frac{qB_o}{2M}$$

as desired.

(6) The energy density of the B-field is

$$\frac{B^2}{2\mu_o} \simeq 4\times 10^{-14} \text{ J/m}^3.$$

The volume of the galaxy (which we'll take to be cylindrical) is roughly

$$V = \pi r^2 h \approx \pi(5\times 10^{20})(10^{19}) \approx 8\times 10^{60} \text{ m}^3,$$

The total energy then is roughly

$$U_B = \frac{B^2}{2\mu_o}V \approx 3\times 10^{47} \text{ J}.$$

Seems large! But with a star with typical power of 10^{37} J/s (in radiation) the magnetic energy is

$$U_B = Pt \implies t \approx 3\times 10^10 \text{ s } \simeq 1000 \text{ years},$$

a short moment in the lifetime of a galaxy. For fun we can also compare to the energy of the gravitational wave that was discovered in 2015, about $3M_{sun}c^2 \simeq 5\times 10^{47}$ J, about the same energy from *one* binary black hole coalecence event! Although of course the magnetic field is not generated by stars or starlight. I wonder where did that B-field energy come from and why is it locked in place?

(7) Solution in the text.

(8) Inside the capacitor we have $E = \sigma/\epsilon_o = Q/A\epsilon_o$. The rate of change of the flux is thus,

$$A\frac{dQ}{dt}\frac{1}{A\epsilon_o} = \frac{I}{\epsilon_o} \text{ and } I_d = I$$

as expected. As for the sign,... As for the electric field it points to the left, since the right plate is positive. But the positive current flows onto the *left* plate, making the derivative $\partial\mathbf{E}/\partial t$ point to the *right*. So this is how the displacement current flows to the right and into the bag-like surface. Closing the surface so that the total surface is S and S' (or what I called S andS_B in class), and has no boundary, we see then that the displacement current flows *in* with magnitude I and the actual current flows *out* with magnitude I. Hence, the net flux vanishes as it must, since the line integral of $\mathbf{B}$ around *no* boundary vanishes.

(9) Because the plates are close together we can neglect the edge field. If so then the electric field between the plates is

$$E = \frac{\sigma}{\epsilon_o} = \frac{Q}{\pi b^2 \epsilon_o}$$

The current density vanishes inside the capacitor $\mathbf{J} = 0$ so Ampere with Maxwell's correct becomes

$$\int_{\partial S} \mathbf{B} \cdot d\mathbf{s} = \mu_o \epsilon_o \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a}.$$

If the loop ∂S points out of the page at the top (where pint P is) and into the page at the bottom then $\mathbf{E}$ and $d\mathbf{A}$ point in the same direction. Meanwhile $dE/dt = I/\pi b^2 \epsilon_o$. So,

$$\mu_o \epsilon_o \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a} = \mu_o \epsilon_o \frac{I}{\pi b^2 \epsilon_o} \pi r^2$$

and

$$\int_{\partial S} \mathbf{B} \cdot d\mathbf{s} = B 2\pi r.$$

This gives

$$B = \frac{\mu_o I r}{2\pi b^2}$$

as expected.

The relation to the earlier figure is that the calculation is mathematically equivalent. Here,

$$B 2\pi r = \mu_o \epsilon_o \frac{d\Phi_E}{dt}$$

while there

$$E 2\pi r = -\frac{d\Phi_B}{dt}.$$

(10) Wireless charging

(a) The devices use Faraday's law - a changing magnetic flux produces an emf which can charge up a battery.

(b)

$$\mathcal{E} = M \frac{dI}{dt} \simeq 9 \text{ V}.$$