Solutions:

(1) To check these solutions we need to check the wave equation and a couple of Maxwell's equations, all of Maxwell's equations in vacuum, or the identities we derived in class. Here I'll do the first

(a) Taking the Laplacian of the electric field $\mathbf{E} = E_o \cos(kz - \omega t)\hat{\mathbf{x}}$

$$\nabla^2 \mathbf{E} = -k^2 \mathbf{E} \text{ while } \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\omega^2 \mathbf{E}$$

so we have a solution to the wave equation

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

if

$$k^2 = \frac{\omega^2}{c^2} \text{ or } \frac{\omega}{k} = c,$$

which holds for these waves. (A similar calculation works for the B-filed.)

Taking the divergence $\nabla \cdot \mathbf{E}$ of the wave solution gives

$$\nabla \cdot \mathbf{E} = -k\hat{\mathbf{z}} \cdot E_o \sin(kz + \omega t)\hat{\mathbf{x}} = 0$$

since $\hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0$. So Gauss' law in vacuum holds. (The "no magnetic monopoles" law $\nabla \cdot \mathbf{B} = 0$ works similarly.)

Both Maxwell's equation ($\nabla \times \mathbf{B} = (1/c^2)\partial \mathbf{E}/\partial t$) and Faraday's equation ($\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$) require $k = \omega/c$. For instance computing

$$\nabla \times \mathbf{B} = -k\hat{\mathbf{z}} \times \frac{E_o}{c}\sin(kz + \omega t)\hat{\mathbf{y}} = k\hat{\mathbf{x}}\frac{E_o}{c}\sin(kz + \omega t)$$

since $-\hat{\mathbf{z}} \times \hat{\mathbf{y}} = \hat{\mathbf{x}}$. Meanwhile,

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = -\frac{\omega}{c^2}E_o \sin(kx + \omega t)\hat{\mathbf{x}}$$

so that Maxwell's equation gives

$$k\frac{E_o}{c}\sin(kz + \omega t) = \frac{\omega}{c^2}E_o \sin(kz + \omega t)$$

or

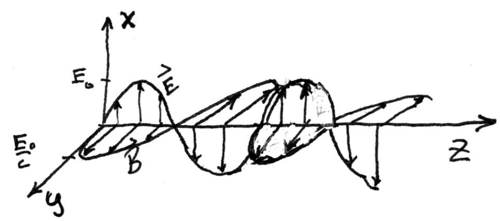
$$k = \frac{\omega}{c}, \text{ as above.}$$

Faraday's equation has the same result.

(b) Since $\hat{\mathbf{k}}$ points in the same direction as the Poynting vector (and the direction of propagation)

$$\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_o} = |S|\hat{\mathbf{z}} \text{ and so } \hat{\mathbf{k}} = \hat{\mathbf{x}}.$$

(c) Here's a rough sketch:



(d) It's in the direction of $\mathbf{E}$ and so $\hat{\mathbf{x}}$.

(2) The left-moving 10^8 Hz wave travels in the $-\hat{\imath}$ direction so

$$\mathbf{k} = -\frac{2\pi}{\lambda}\hat{\imath} = -\frac{2\pi f}{c}\hat{\imath} \simeq -2.09\ \mathrm{m}^{-1}\hat{\imath}.$$

while $\omega = c/k \simeq 6.28 \times 10^{10}\ \mathrm{s}^{-1}$. Since $\mathbf{E} \times \mathbf{B}$ is parallel to $\mathbf{k}$ this structures the directions. We don't have a phase of the wave... I chose $\mathbf{E}$ in the $\hat{\jmath}$ direction so $\mathbf{B}$ points in the negative z-direction. Assembling these results we have

$$\mathbf{E} = E_o\hat{\jmath}\cos(kx + \omega t). \text{ and } \mathbf{B} = -B_o\hat{k}\cos(kx + \omega t)$$

where ω and k are given above. The last bit is the amplitude of the B-field - it is E_o/c as we saw in class, on Friday :(So, finally

$$\mathbf{E} = E_o\hat{\jmath}\cos(kx + \omega t). \text{ and } \mathbf{B} = -\frac{E_o}{c}\hat{k}\cos(kx + \omega t).$$

Your correct answer might differ from this: You could choose another direction of the electric field, i.e. $-\hat{\jmath}$. You could choose sine instead.

(3) Light's momentum

(a) We already found the average *energy* density carried by plane, monochromatic waves

$$\langle S \rangle = \frac{1}{2} c\epsilon_o E_o^2.$$

Since the expressions are multiples of the Poynting vector, we can just use that here for the momentum density,

$$\langle \mathcal{P} \rangle = \frac{\langle S \rangle}{c^2} = \frac{1}{2c}\epsilon_o E_o^2,$$

still proportional to the amplitude squared.

(b) For the solar sail we need,

$$F = \frac{\Delta p}{\Delta t} = \frac{1}{2} mg$$

so with the total change in momentum from the reflected light

$$\Delta p = 2\langle \mathcal{P} \rangle Ac\Delta t,$$

where A is the area of the sail. With $S = 1360\ \mathrm{W/m^2}$ we have

$$\frac{\Delta p}{\Delta t} = 2\frac{\langle S \rangle}{c}A \implies A = \frac{mgc}{4\langle S \rangle} \simeq 1.2 \times 10^9\ \mathrm{m}^2 \simeq 1.2 \times 10^3\ \mathrm{km}^2$$

Yikes! That's a large sail. Well, I did ask for a large acceleration... Maybe not the best choice for propulsion.

(4) The area is $A = \pi R^2$ so the intensity (and magnitude of the Poynting vector) is

$$S = \frac{P}{A} \simeq 7.8 \times 10^{-8} \mathrm{W/m}^2$$

Since $S = \bar{E}^2/377\,\Omega$, the 'rms' value of the electric field is

$$\bar{E} \simeq 0.002 \text{ V/m}.$$

Or, directly,

$$\bar{E} = \sqrt{\frac{377 \cdot P}{\pi R^2}} \simeq 0.0022 \text{ V/m}.$$

(5) Solution in text.

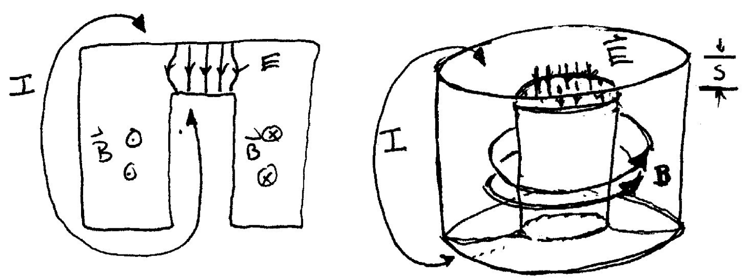
(6) This microwave cavity has a capacitor at the top and an inductor around the cylinder to make the LC circuit. The magnetic field wraps around the inner cylinder. Since this is an LC circuit we know that the natural angular frequency is $\omega_o = \sqrt{1/LC}$. The inductance is

$$L = \frac{\mu_o h}{2\pi} \ln \frac{b}{a}$$

from equation (7.62) while the capacitance is the usual $\epsilon_o A/s = \epsilon_o \pi a^2/s$ for parallel plate capacitors. Cool. Putting these two things together gives the result

$$\omega_o = \frac{c}{a}\sqrt{\frac{2s}{h\ln(b/a)}}$$

where $c^2 = 1/\mu_o\epsilon_o$, as we know. Here are a couple of sketches of the fields and current.



A clever little thing.