Electromagnetism (PHYS 295): Solutions 8

Spring 2025 v1.0

Solutions:

(1) This one is done in the book. Be sure to include all the steps in your solution.

(2) Computations of curl and divergence.

(a) Taking the curl

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & -x+y & -2z \end{vmatrix}$$
$$= \hat{\imath}(0) + \hat{\jmath}(0) + \hat{k}(-2) = -2\hat{k}$$

Meanwhile the divergence is

$$\nabla \cdot \mathbf{F} = 1 + 1 - 2 = 0.$$

Since $\nabla \times \mathbf{F} \neq 0$ there is no ϕ for $\mathbf{F}$.

(b) Taking the curl

$$\nabla \times \mathbf{G} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 2x+3z & 3y \end{vmatrix}$$
$$= \hat{\imath}(3-3) + \hat{\jmath}(0) + \hat{k}(2-2) = 0.$$

Meanwhile the divergence is

$$\nabla \cdot \mathbf{G} = 0.$$

Since $\nabla \times \mathbf{G} = 0$ there is a ϕ for $\mathbf{G}$ such that $\mathbf{G} = -\nabla\phi$. Taking the x-component, $2y$ and integrating we have, to start, $\phi = -2xy + f(y,z)$. Similarly for the y-component $2x+3z$, implying $\phi = -2xy - 3xy + g(x,z)$. So it looks like $\phi = -2xy - 3xy + C$. Checking this shows that indeed $\mathbf{G} = -\nabla\phi$.

(c) Taking the curl

$$\nabla \times \mathbf{H} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - z^2 & 2 & 2xz \end{vmatrix}$$
$$= \hat{\imath}(0) + \hat{\jmath}(-2z - 2z) + \hat{k}(0) = -4\hat{\jmath}.$$

Meanwhile the divergence is

$$\nabla \cdot \mathbf{H} = 4x.$$

Since $\nabla \times \mathbf{H} \neq 0$ there is no ϕ for $\mathbf{H}$.

(3) (a) Computing the cross product

$$q\vec{v} \times \vec{B} = q\left[\left(-0.1\hat{\imath} + 2\hat{\jmath} - 6\hat{k}\right) \times 10^5\right] \times \left[0.05\hat{\jmath} + 0.1\hat{k}\right] = q\left(0.5\hat{\imath} + 0.01\hat{\jmath} - 0.05\hat{k}\right) \times 10^5.$$

So

$$\vec{F} = \left(80\hat{\imath} + 1.5\hat{\jmath} - 8\hat{k}\right) \times 10^{-16}\ \text{N}.$$

(b) Solved in text

(4) Recall that the B-field inside a long solenoid is $B = \mu_o nI$. Outside the solenoid the field vanishes. With these results we can use superposition to obtain the field in this configuration. When we consider $r > b$ we are outside both solenoids so $B = 0$. In between, $b < r < a$, we are inside the outer solenoid and outside the inner one so $B = \mu_o n_2 I$. This field points to the right (by a right hand rule). Inside, $r < a$, we're inside both solenoids so there are contributions from both. Given the change in the current flow on the inner solenoid, we have $B = \mu_o(n_2 - n_1)I$ in the rightwards direction. The actual direction of B will depend on the relative size of n_2 and n_1.

This can be solved using Amperé's law arguments as well.

(5) This is a really big γ, $\gamma \sim 10^{10}$![1] The angular size of the pancake (actually more like 'pan-paper' - it is thin) is $\Delta\theta \sim 1/\gamma$. The maximum field, on the z axis perpendicular to the velocity, is

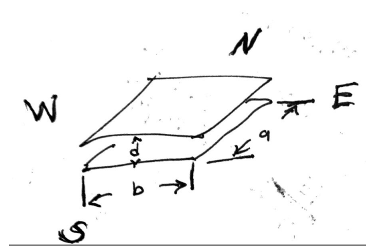
$$E_{max} = \gamma \frac{Q}{4\pi\epsilon_o} \frac{1}{r^2}$$

This is from the E-field we derived in class. Setting this to 1 V/m gives a radius of

$$r = \sqrt{\frac{e\gamma}{4\pi\epsilon_o}} \simeq 3.8 \text{ m}.$$

Gracious, that is very far from the proton! The thickness of the pancake at this radius is $r\Delta\theta = r/\gamma \simeq 3.8$ Å. That is thin. The particle is essentially moving at the speed of light and this field of 1 V/m would last about 1×10^{-18} s, a very short interval of time!

(6) A question of many questions! Here's a quick sketch of the geometry I'll use



Parallel plate capacitors satisfy

$$C = \frac{\epsilon_o A}{d} = \frac{\epsilon_o ab}{d} \text{ and } Q = CV$$

Breaking it down into the obvious parts...

(a) The electric field strength is

$$E = \frac{\Delta V}{d} = 1.5 \times 10^4 \text{ V/m} = E_\perp$$

(Alternatively as we saw in class

$$E = \frac{\sigma}{\epsilon_o} = \frac{Q}{\epsilon_o A} = \frac{V}{d})$$

[1] By the way these protons (or other nuclei) actually exist and have a number of astrophysical puzzles associated to them, e.g. How are they accelerated to such high energies? They are observed at cosmic ray observatories like Auger in South America and LHAASO in China.

(b) The number of electrons N_e is

$$N_e = \frac{Q}{e} = \frac{\epsilon_o abV}{ed} = \frac{\epsilon_o V}{e} \simeq 1.66 \times 10^{10}.$$

(David Morin arranged for the numbers to work out so that $ab/d = 1$.)

(c) Now we're asked for what happens when the plate moves east at $0.6c = 3/5c$. It is useful to compute γ for what follows

$$\gamma = \frac{1}{\sqrt{1 - \frac{9}{25}}} = \frac{5}{4}.$$

Again as we saw in class only b and E are affected in this frame. So

$$\begin{aligned}
a' &= a = 10 \text{ cm} \\
b' &= \frac{b}{\gamma} = 16 \text{ cm, due to length contraction} \\
d' &= d = 2 \text{ cm} \\
N'_e &= N_e \text{ due to charge invariance} \\
E'_\perp &= \gamma E_\perp \simeq 1.88 \times 10^4 \text{ V/m}
\end{aligned}$$

(d) If the velocity is directed upward then

$$\begin{aligned}
a' &= a = 10 \text{ cm} \\
b' &= b = 20 \text{ cm} \\
d' &= \frac{d}{\gamma} = 1.6 \text{ cm, due to length contraction} \\
N'_e &= N_e \text{ due to charge invariance} \\
E'_\parallel &= E_\parallel = 1.5 \times 10^4 \text{ V/m}
\end{aligned}$$

(7) As we saw in class the electric field is directed along the radius. So let's choose a sphere centered on the charge. Then $\mathbf{E} \cdot d\mathbf{a} = E\, da$. I'll use the book's notation for this "prime" frame so angles on the sphere are θ' and φ'. The flux is then

$$\int \mathbf{E} \cdot d\mathbf{a} = \int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \frac{Q}{4\pi\epsilon_o\, r'^2} \frac{1 - v^2}{\left(1 - v^2 \sin^2\theta'\right)^{3/2}} r'^2 \sin\theta'.$$

I've set $c = 1$. Canceling a factor of r'^2, doing the φ integration, and cleaning up gives

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{Q(1 - v^2)}{2\epsilon_o} \int_0^{\pi} d\theta' \frac{1}{\left(1 - v^2 \sin^2\theta'\right)^{3/2}} \sin\theta'.$$

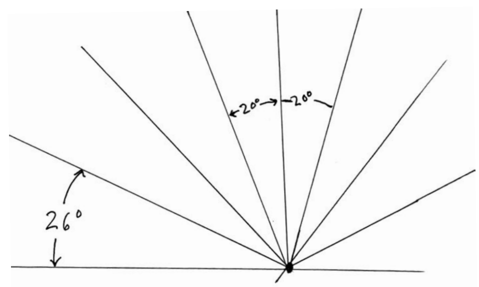
This integral is in Appendix K. The formula K.15 gives

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{Q(1 - v^2)}{2\epsilon_o} \left[\frac{-\cos\theta'}{(1 - v^2)\left(1 - v^2 \sin^2\theta'\right)^{1/2}} \right]_0^{\pi} = \frac{Q}{\epsilon_o},$$

as expected.

(8) So much in a picture!

(a) Let's work from outside to inside. To understand the outside field lines I used a ruler to trace back the "early" fields lines to the x-axis. These field lines would have intersected the axis at $x = 12$ cm, if there had not been acceleration. The density of these lines is *not* uniform (!)



They are a bit "pancaked". To determine how much, I measured the angles as shown. Since the density of field lines is proportional to the strength of the field we can compare the field at $\pi/2$ with the field at 0 (or π). The expression to use is equation (5.15). Thus, from the equation,

$$\frac{E'|_{\pi/2}}{E'|_0} = \gamma,$$

while from the angles

$$\frac{E'|_{\pi/2}}{E'|_0} = \frac{\Delta\theta'|_{\pi/2}}{\Delta\theta'|_0} \simeq \frac{0.454}{0.349} = 1.3$$

where I converted the angular spacing $\Delta\theta'$ above to radians. So $\gamma \simeq 1.3$. From this

$$v = \sqrt{1 - 1/\gamma^2} \simeq 0.769,$$

or pretty much $0.8c$. If there had been no acceleration the electron would be at 12 cm. Proceeding inwards, the next feature is those sharp turns or angles in the field lines. They have an outer radius of about 16 cm and an inner radius of about 14 cm. This is caused by acceleration. Since the news of the change in field lines can only travel as fast as c, this acceleration occurred between $t_1 = r_1/c = (14 \times 10^{-2})/3 \times 10^8 \simeq 0.47$ ns and $t_2 = r_2/c = (16 \times 10^{-2})/3 \times 10^8 \simeq 0.53$ ns before $t = 0$. The inside lines are symmetric and come from the origin. Therefore, the charge is now (at $t = 0$) at rest.

To summarize, the electron was moving to the right at $v = 0.8c$. It slowed to rest between 0.53 ns and 0.47 ns before $t = 0$.

Notice that you can also get an estimate for the speed of the electron from the back-tracing position at 12 cm divided by the time at the middle of the acceleration at (15 cm)$/c$ or $v = 12c/15 = 0.8c$.

(b) Since the electron came to rest at $t = -0.47$ ns, at $t = -0.75$ ns the charge was still moving. Assuming it suddenly came to rest at -0.5 ns (midway between the t_1 and t_2, it would have been at $x = 0.8c \cdot 0.5$ ns $= -6$ cm at -0.75 ns. (You can obtain the identical result with more work by assuming a linear deceleration and summing up the distance the electron traveled when it was at $0.8c$ and the "$1/2at^2$" distance it traveled as it slowed down.)

(c) Using equation (5.15) at $\theta' = \pi$ then

$$E' = \frac{e}{4\pi\epsilon_o}\frac{1 - v^2}{r'^2} \simeq 1.4 \times 10^{-7} \text{ V/m}$$

with $r' = -6$ cm and $v = 0.8$.

(9) (a) By Gauss's law for a cylindrical surface around the rod, the electric field is by $\lambda/2\pi\epsilon_o r$. This is the result in the frame shown in figure 5.30. Boosting to the frame of the particle gives $\gamma\lambda/2\pi\epsilon_o r$, since the field is transverse and since r doesn't change in the transformation. The force is larger in the particle's rest frame. (This provides another way to solve this problem.)

(b) There is now a non-vanishing B-field but the force vanishes since we are in the rest frame of the particle. The electric field has increased due to length contraction and we obtain the same result as above, $\gamma\lambda/2\pi\epsilon_o r$, as we must.

(10) To achieve the required uniformity we need to ask that the superposition of the fields from the two coils yields a region in which the B-field is approximately constant. Placing one coil at $z = b/2$ and one coil at $z = -b/2$ we have the z component

$$B_z(z) = \frac{\mu_o I a^2}{2(a^2 + (z + b/2)^2)^{3/2}} + \frac{\mu_o I a^2}{2(a^2 + (z - b/2)^2)^{3/2}}$$

Each one of these fields is most intense in the plane of the ring so the idea is to place the two rings so that the peak is flattened out, producing a constant field. Taking a Taylor series around $z = 0$ and dropping distracting constants yields

$$B_z(z) - B_z(0) \propto \left\{ 3\, \frac{5(z + b/2)^2 - [a^2 + (z + b/2)^2]}{[a^2 + (z + b/2)^2]^{7/2}}\Bigg|_{z=0} + \frac{5(z - b/2)^2 - [a^2 + (z - b/2)^2]}{[a^2 + (z - b/2)^2]^{7/2}}\Bigg|_{z=0} \right\} z^2$$

as the first (and all odd orders) derivative vanishes. The two terms in the curly brackets simplify to

$$\frac{3(b^2 - a^a)}{(a^2 + b^2/4)^{7/2}},$$

which vanishes when $b = a$. If so then

$$B_z(0) = \sqrt{\frac{2}{5^3}}\, \frac{\mu_o I}{a}$$

and the first correction term is at fourth order in z. So if you are halfway between the top coil and the origin, at $z = b/4$, then your local B-field only differs from the field at the center of the coils by $1/4^4 \simeq 0.04\%$. So if you move closer to the center just a bit you can achieve uniformity within a part in a thousand - in the z direction. The transverse problem is harder...

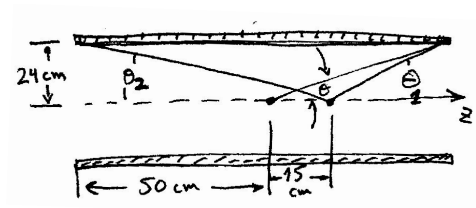
(11) To cancel a tilted 0.55 gauss field with a solenoid we just need to add an equal and opposite field. Seems pretty simple. We enclose the 30^3 cm^3 region inside a long solenoid (so we don't have to worry about edge effects) with $r \simeq 24$ cm ($r > a/\sqrt{2} \simeq 11$ cm, the dimension along the diagonal) solenoid tilted by 30^o from the vertical. We need a current density (nI) given by

$$B = \mu_o n I = 5.5 \times 10^{-5}\ \text{T, or } nI \simeq 44\ \text{A/m}$$

So... why those three stars .... ?? Oh blast, this isn't an infinitely long solenoid! The actual field is given by equation (6.56).The design specifications call for no more than 10 milligauss of deviation of $10^{-2}/0.55 \simeq 0.018$. Comparing the B-field at the center with the B field 30/2=15 cm away gives

$$\frac{B_z(0)}{B_z(15)} = \frac{2\cos\theta}{\cos\theta_1 + \cos\theta_2}$$

where the angles are shown in the figure. (I defined θ_2 differently than in the text. The change results in the cosine of θ_2 changing sign. Physically the results are identical.)



I set up a spreadsheet to compute the ratio of the fields. With these dimensions I got a 2% deviation which is larger than the 1.8 % maximum, bummer. After playing around with this I find that it is easiest to reduce the ratio by increasing the solenoid's length. With a radius of 23 cm (so the box fits with a little extra room) and a length of 1.25 m gives a deviation of 1 %, which I think should be safe. Here a snapshot:

	A	B	C	D	E	F	G	H
1	length (cm)	radius (cm)	edge of region		Required Box (cm)		min radius (cm)	
2	125	23	15		30		21.2132034	
3								
4	angles							
5	theta	0.35261961		cos theta	0.93847123			
6	theta1	0.45093643						
7	theta2	0.28849472						
8								
9	Bz(0)	1.87694247						
10	Bz(edge)	1.85871262						
11								
12	ratio of B's	1.00980778						
13								

Returning to the B-field to correct the previous calculation

$$B = \mu_o n I \cos\theta \implies nI \simeq 47 \text{ A/m, or over 1.25 m, 61 A-turns}$$

Your numerical answers may differ from these results. This solution only considers the field on the z axis so a more careful analysis would include the magnetic field off the axis.

The figures on page 301 also are a way in which you can solve this graphically

In practice one could shorten the coil further by adding additional "trim coils" on the ends of the solenoid to reduce the non-uniformity of the field at the ends. But that problem would be like a superposition of the last two problems!