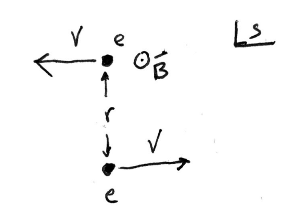
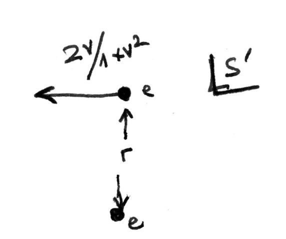
Solutions:

(1) The reasoning here will be similar to what we did in class: We'll compute the force in one frame and then transform back to see what $F = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ gives for B in the original frame. I'll call the 'lab frame' S. It looks like this



I anticipated that the new B will be out of the page - more on this soon. Let's set $c = 1$. I chose to switch to the rest frame of the proton below. So moving right at v we enter the S' frame which looks like this



The lower proton is at rest in S'. Notice how I included the new velocity from relativistic velocity addition.

From our work before, we know that the electric field is focused by γ' in the vertical direction, where γ' is the dilation factor for the *relativistically added* velocity,

$$\begin{aligned}\gamma' &= \frac{1}{\sqrt{1 - \left(\frac{2v}{1+v^2}\right)^2}} \\ &= \frac{1+v^2}{\sqrt{1+2v^2+v^4-4v^2}} \\ &= \frac{1+v^2}{1-v^2}\end{aligned}$$

where I expanded the velocity and collected terms. So the electric field in the S' frame is

$$E_{S'} = \gamma' \frac{e}{4\pi\epsilon_o r^2} = \frac{e(1+v^2)}{4\pi\epsilon_o (1-v^2) r^2}.$$

(Just a reminder that since the radii are perpendicular to the velocity, there is no length contraction, $r' = r$.) The force in S' is then

$$F_{S'} = eE_{S'} = \frac{e^2(1+v^2)}{4\pi\epsilon_o (1-v^2) r^2}.$$

This completes our work in the S' frame. (There is no force from the magnetic field since in this frame the lower charge is at rest.)

You can complete the problem but finding the E and B fields in the frame S. But, beware, we cannot use the Biot-Savart law since that only applies to stationary currents. I'll instead use forces as discussed in section 5.8.

Switching back to the lab frame S by moving left by v we have

$$F_{S'} = \gamma F_S$$

where γ is the usual $1/\sqrt{1-v^2}$ and

$$F_S = q(\mathbf{E}_S + \mathbf{v} \times \mathbf{B}_S).$$

Let's put in what we know - and the suggestion in the problem that $B_S = vE_S$. I have indicated the direction of the B field in the first diagram. This ensures that we have the expected repulsion via $qv \times B$. Hence, γF_S is

$$\begin{aligned}
\gamma F_S &= e\frac{1}{\sqrt{1-v^2}}\left(E_S + vB_S\right) \\
&= e\frac{1}{\sqrt{1-v^2}}\left(\frac{e\gamma}{4\pi\epsilon_o r^2} + v^2\frac{e\gamma}{4\pi\epsilon_o r^2}\right) \\
&= e^2\left(\frac{1}{1-v^2}\right)\left(\frac{1}{4\pi\epsilon_o r^2}\right)\left(1+v^2\right) \\
&= \frac{e^2(1+v^2)}{4\pi\epsilon_o(1-v^2)r^2} \\
&= F_{S'}!
\end{aligned}$$

as above. It works!

(2) Solution in the text.

(3) Answer in the text. You can also solve this using the transformation of a current carrying wire, see the argument starting on page 259 and in class notes.

(4) The B-fields produced by the wires are clockwise for A and C and counterclockwise for B. And, by Amperé's law they have the form

$$B = \frac{\mu_o I}{2\pi r}$$

So in the center the contributions of A and C cancel. The B-field is then

$$B_{center} = \frac{\mu_o 2I}{2\pi r} = \frac{\sqrt{2}\mu_o I}{\pi d}$$

directed diagonally down- and left-ward.

On the bottom right corner all three wires contribute. Wire A has a B field of

$$B_A = \frac{\mu_o I}{2\pi d} \text{ to the right.}$$

Wire C has a B field of

$$B_C = \frac{\mu_o I}{2\pi d} \text{ upwards.}$$

The vector sum of these two gives a B-field of $\mu_o I/\sqrt{2}\pi d$ up and to the right. But wire B has a B field of

$$B_{=}\frac{\mu_o 2I}{2\pi r}\frac{\sqrt{2}\mu_o I}{\pi 2d} \text{ diagonally down- and left-ward.}$$

So the total sum vanishes.

(5) The rectangular loop derivation we did in class is fine but here is a slightly more general result here. Let's consider a loop in the xy plane. The z component of the B-field is not going to produce torque around the x or y axes. To start let's look at the z component of the force on a wee length $d\ell$ of the loop, as drawn in figure 6.41

$$dF_z = I(d\ell \times \mathbf{B})_z = Id\ell B_y \sin\theta = IdxB_y$$

where the last equality is due to the geometry; the angle θ is between the y direction and $d\ell$ and $d\ell \sin\theta$ is just dx. The torque on this bit is $d\tau_x = yIdxB_y$. Integrating up we have

$$\tau_x = \int d\tau_x = IB_y \int ydx = IaB_y,$$

where $a = \int ydx$ is the area of the loop. Now, you might be wondering about the other component τ_y. Following the same argument as above the torque τ_y contains the integral $\int xdx$, which vanishes so the x component is the only surviving bit, so $\vec{\tau} = IaB_y\hat{\imath}$. Using the magnetic moment $\vec{\mu}$

$$\vec{\mu} \times \mathbf{B} = -Ia\hat{\mathbf{k}} \times (B_x\hat{\imath} + B_y\hat{\jmath}) = IaB_y\hat{\imath}$$

as above, so $\vec{\tau} = \vec{\mu} \times \mathbf{B}$.

The net force of the loop

$$\oint d\mathbf{F} = I \oint d\ell \times \mathbf{B} = 0$$

since $\oint d\ell = 0$ due to the loop being a loop.

(6) Superposition! Let's first calculate the B-field of a wire with uniform current density J and radius a. Using a cross-sectional Ampèrian loop of radius r inside the wire we have

$$\oint \vec{B} \cdot d\vec{\ell} = B2\pi r$$

while the current enclosed is

$$I_{encl} = I\frac{\pi r^2}{\pi a^2}. \text{ Hence, } \vec{B} = \frac{\mu_o I r}{2\pi a^2}\hat{\phi}.$$

On the outside the calculation is similar except the loop contains the full current I,

$$\vec{B} = \frac{\mu_o I}{2\pi r}\hat{\phi}.$$

On the boundary when $r = a$ we have

$$\vec{B} = \frac{\mu_o I}{2\pi a}\hat{\phi}.$$

Ok to build the configuration show we'll take a wire with radius $b = 4$ cm with current flowing into the page and *add* a wire with radius $a = 2$ cm and current flowing *out* of the page. We also have to calculate the "I" since we are given the amount of current in the crescent-shaped wire. Let's start with the current density for the crescent,

$$J = \frac{I_c}{A} = \frac{900 \text{ A}}{\pi b^2 - \pi a^2} \text{ so in the larger wire } I_b = \pi b^2 J = \frac{900 \text{ A}}{12} \cdot 16 = 1200 \text{ A}.$$

and in the smaller radius wire,

$$I_a = \pi a^2 J = \frac{900 \text{ A}}{12} \cdot 4 = 300 \text{ A}.$$

At P, the center of the big wire, the only field is due to the smaller wire, so

$$\vec{B} = -\frac{\mu_o I}{2\pi a}\hat{\phi} = -\frac{\mu_o 300}{2\pi 0.02 \text{ m}}\hat{\phi} = 30 \text{ gauss},$$

pointing leftwards.

(7) Your superposition alarms should be ringing. We'll find the field by adding up the contributions of the semi-circle and the two lines of current. As you can check with the right hand rule, all contributions add due to the direction of the current flow. The semi-circle gives half of the B-field along the z-axis of a ring of current evaluated at $z = 0$. This is

$$B = \left.\frac{\mu_o I r^2}{2(r^2 + z^2)^{3/2}}\right|_{z=0} = \frac{\mu_o I}{2r}.$$

Each wire gives half of the field of a long wire so the total is

$$B = \frac{1}{2}\frac{\mu_o I}{2r} + 2 \cdot \frac{1}{2}\frac{\mu_o I}{2\pi r} = \left(\frac{1}{\pi} + \frac{1}{2}\right)\frac{\mu_o I}{2r} \simeq 0.409\frac{\mu_o I}{r}$$

(If you want to find these fields directly you can use the Biot-Savart law, but I would not recommend it.)

(8) Treating the source of earth's field as a conducting ring of radius b we have

$$B = \frac{\mu_o I b^2}{2(b^2 + z^2)^3/2}$$

at z on the $N - S$ pole axis. Given the assumption that $b = R/2$ and at the pole $z = R$ then we find after some algebra

$$B = \frac{\mu_o}{5\sqrt{5}R}.$$

So if $B = 0.5$ gauss or 5×10^{-5} T then $I \simeq 2.7 \times 10^9$ A, which is a mammoth current. Lighting strikes have current on the order of 10^5 A. The earth's field is more likely produced by a set of solenoid-like flows between in the solid core and the earth's mantel. Curiously, I don't think that this problem of the origin of planetary magnetic fields is completely understood.

(9) The wire has an oscillating position of $x(t) = x_o \sin(\omega t + \varphi)$ (since we are eventually only interested in the maximum, cosine would also be fine). The change in flux is then

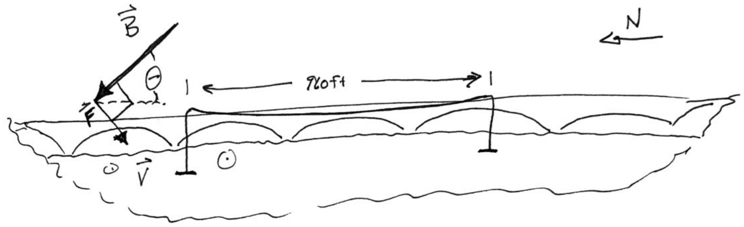
$$\frac{d}{dt}\int_s \vec{B} \cdot d\vec{a} = B\ell\frac{dx}{dt} = B\ell x_o \omega \cos((\omega t + \varphi))$$

with a maximum value of $B\ell x_o \omega$. With $\omega = 2\pi f$ this gives the maximum emf, $\mathcal{E} \simeq 0.034$ V.

(10) Faraday's experiment was a difficult one. Ions in the brackish water flow past the bridge and the $qv \times B$ force drives a current around a loop. But it is difficult to detect. Here's why:

The force per unit charge (the electric field) is

$$\frac{\vec{F}}{q} = \vec{v} \times \vec{B} = vB \text{ directed as shown}$$



The force that pushes the current is the horizontal component so from the geometry above

$$\frac{F_{\vec{circuit}}}{q} = vB\cos(90 - \theta) = vB\sin\theta.$$

As stated in the prob;em notes $\theta = 66.5°$. Between the two ends of the line, $\ell = 960$ ft $\simeq 293$ m apart this produces a voltage of

$$V = vB\ell\sin\theta \simeq 20 \text{ mV}.$$

which was too small for Faraday's equipment to measure.