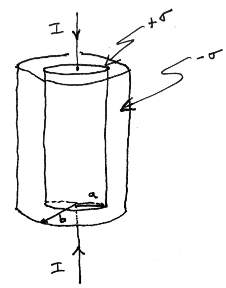
Electromagnetism (PHYS 295): Last Question Spring 2025

In our last we considered a charging cylindrical capacitor with σ_a charge density on the inner cylinder and σ_b on the outer cylinder



Compared to the picture in class I have added current sources to the inner cylinder. (We'll see why at the end. The outer cylinder would have some wires too. But I have not shown them.) $\sigma_a > 0$ and $d\sigma_a/dt > 0$.

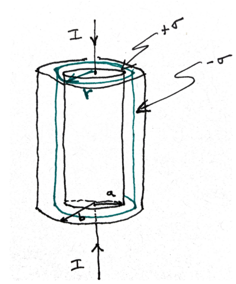
As in class we can derive an electric field around the inner cylinder using Gauss' law

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{encl}}{\epsilon_o}.$$

Due to the accumulating positive charge on the inner cylinder the electric field is radial. The flux through a surface at radius r and height ℓ is

$$\int \mathbf{E} \cdot d\mathbf{a} = E 2\pi r \ell.$$

Here's a sketch of the gaussian surface,



Meanwhile the charge enclosed is

$$Q_{encl} = \sigma_a 2\pi a \ell = Q.$$

Thus,

$$\mathbf{E} = \frac{\sigma_a a}{\epsilon_o r} \hat{r}.$$

Now what about a magnetic field? The electric flux through the green surface

$$\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{a} = \frac{\sigma_a 2\pi a \ell}{\epsilon_o}$$

is increasing at a rate of

$$\frac{d\Phi_E}{dt} = \frac{d}{dt} \int_S \mathbf{E} \cdot d\mathbf{a} = \frac{d\sigma_a}{dt} \frac{2\pi a \ell}{\epsilon_o} = \frac{1}{\epsilon_o} \frac{dQ}{dt} > 0.$$

So it looks like the magnetic field wraps around the top and bottom of the cylinder, just like you would expect based on the current flows.

The boundary of the green surface S consists of the two loops at the top and bottom of the cylinder - the green circles in the sketch. Since the area element points outward, the orientation on the top loop is clockwise and on the bottom loop is counter clockwise. Hence, Maxwell's equation gives

$$\oint_{\partial S} \mathbf{B} \cdot d\mathbf{s} = \mu_o \epsilon_o \frac{d}{dt} \int_S \mathbf{E} \cdot d\mathbf{a} = \mu_o \frac{dQ}{dt}.$$

The left hand side splits into top and bottom loops,

$$\oint_{\partial S} \mathbf{B} \cdot d\mathbf{s} = \int_{top} \mathbf{B} \cdot d\mathbf{s} + \int_{bottom} \mathbf{B} \cdot d\mathbf{s} = 2B \cdot 2\pi r$$

So the magnitude of the B-field is

$$B = \frac{\mu_o}{4\pi r} \frac{dQ}{dt}.$$

So that's the solution of the last question.

A few comments:

- We can also write the solution as

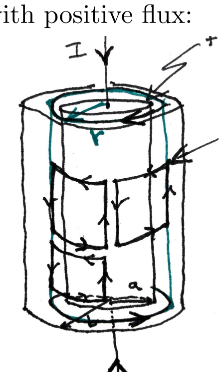
  $$B = \mu_o \frac{d\sigma_a}{dt} \frac{a\ell}{2r}$$

  The existence of the ℓ seemed bothersome in the long cylinder case but what it was really telling us is that it takes a large amount of current to make increase the uniform charge in a long cylinder.
- Because current flows from top and bottom $dQ/dt = 2I$ the result can also be written as

  $$B = \frac{\mu_o}{4\pi r} 2I = \frac{\mu_o}{2\pi r} I,$$

  which is the normal result for a current carrying wire.
- You can run the "blown out surface" argument that we used to discover the displacement current in this case too. Here, though, we have to have some current running up and down the cylinder to distribute the uniform charge. The current vanishes at the center of the cylinder. Due to this current flow, the B-field is maximum at the ends and then decreases to zero at the middle of the cylinder.

  The changing electric flux doesn't contribute B-field along the length of the cylinder as you can see by drawing smaller surfaces with positive flux:

  The neighboring induced magnetic fields point in opposite directions, canceling out.
- Finally, one can imagine a substance between the cylinders that separates charge. If so then there is a current running from outer cylinder to inner cylinder. But let's leave this to another day.