Dirac Notation Introduction

Spring 2019 v3.0

1. Braket (Dirac) Notation

Dirac introduced a very beautiful way of expressing the vectors used in quantum mechanics. This is a short introduction to this "bra-ket notation" from the point of view of vector calculus and linear algebra. For those wanting a clean, logical presentation I know of no better than Dirac's, *The Principles of Quantum Mechanics* sections 6-20. What follows is a brief introduction that focuses on basic definitions and vector operations.[1]

Basic idea: A "ket" $|\cdot\rangle$ is a vector.

The components may be complex, hence the space of kets is a *complex vector space*. To keep track of *which* vector we add a label, replacing $\cdot$ above, with some description. For instance, $|+z\rangle$ might represent the "spin up in the z direction" vector. Another example are wavefunctions, since (suitable) functions can be seen to form a vector space, we can write, e.g. $|\psi\rangle$.

Slogan: "Put what you know in the ket." In quantum mechanics, you identify the vector by the last measurement on the system. Suppose you have a particle in a box. If you observed the particle on the left hand side, say $0 < x < L/2$, then the ket would be $|0 < x < L/2\rangle$.

Basis: If you have N basis vectors $|i\rangle, i = 1, 2, \ldots, N$ then any vector $|v\rangle$ is written as

$$|v\rangle = \sum_i v_i\, |i\rangle.$$

It can also be arranged in a column

$$|v\rangle \to \begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_N \end{pmatrix}$$

These expressions are the analog of the usual

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

and

$$\vec{a} \to \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

in the familiar 3D vector space.

Bra: A "bra" $\langle\cdot\,|$ is "dual" to a vector which means that, with a ket, the bra gives a complex number, $\langle\cdot\,|\,\cdot\rangle \in \mathbb{C}$. The bra is an *adjoint* of the vector,

$$\langle a\,| = (|\,a\rangle)^\dagger.$$

[1] The slogans are largely from Chester's *Primer of Quantum Mechanics*, a classic and quirky introduction to quantum mechanics.

The dagger † is the usual notation for adjoint. The mechanics of the adjoint take kets to bras and the components to their complex conjugates. For instance, if

$$\mid neatket\rangle = \frac{i}{\sqrt{2}} \mid 3\rangle$$

then

$$\langle neatbra \mid \equiv (\mid neatket\rangle)^\dagger = \frac{-i}{\sqrt{2}}\langle 3 \mid$$

See how the ket switched to a bra and the number became its complex conjugate? The label on the state does not change. If you are using the column notation for the kets you make a row vector under the adjoint so

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}^\dagger = (v_1^* \; v_2^* \; v_3^*)$$

This all agrees nicely with the linear algebra conventions. Note that the adjoint is the "complex transpose" in that context. This operation is sometimes also called the "Hermitian conjugate".

Inner Product: The scalar product or *inner product* is written as $\langle \cdot \mid \cdot \rangle$. This has many other interpretations as well. The most important interpretation in quantum mechanics is (fanfare!)

The inner product is the probability amplitude.

So this is the beastie that gives predictions! If you have a state $\mid \psi\rangle$ and what to find out whether the spin is up in the z direction then you calculate the square modulus of the probability amplitude,

$$|\langle +z \mid \psi\rangle|^2 = \langle +z \mid \psi\rangle\langle +z \mid \psi\rangle^*$$

and that is the *probability*, (assuming the state $\mid \psi\rangle$ is normalized).

The component, or representation, of a ket vector $\mid a\rangle$ in the basis $\mid i\rangle$ is

$$(\mid i\rangle)^\dagger \mid a\rangle = \langle i \mid a\rangle$$

So

$$\mid a\rangle = \sum_i a_i \mid i\rangle = \sum_i \mid i\rangle\langle i \mid a\rangle$$

In quantum mechanics, you can write a wavefunction $\psi(x)$ as the wavefunction in the x representation, i.e. $\langle x \mid \psi\rangle$.

We usually work with orthonormal bases so that

$$\langle v_i \mid v_j\rangle = \delta_{ij}.$$

You can now write 1 in a new way

$$\sum_i \mid i\rangle\langle i \mid = 1!$$

This states that the basis $\mid i\rangle$ is complete.

Operators: Operators, often written with hats, ˆ (in polite company), take a ket and produce another ket

$$\hat{Q} \mid a\rangle = \mid b\rangle.$$

You can express any operator as a matrix operation by working in a basis like $\mid i\rangle$. This is called "matrix mechanics" (which was discovered by Heisenberg, Born, and Jordan). The operator is entirely determined by how it acts on every basis vector

$$\hat{Q} \mid i\rangle = \sum_j Q_{ij} \mid j\rangle$$

In Math Methods we use this notation for solutions to initial or boundary value problems so the operators are typically differential in flavor.