Mathematical Physics (PHYS 320): Final QPS

Spring 2019 v1.1

Welcome to the final problem set!

- Please submit your solutions by 5 PM, Friday May 17.
- Please check your results.
- Please use your notes, Mathematica, Wolfram Alpha, and Boas, but no other resources. Include printouts of your work if you work with the software.
- Please *cite any references* (source, page number and formula number, as appropriate).
- You may not consult any other resources such as the internet.
- Your solutions must be entirely your own work.
- Please check your results!

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(1) (10 pts.) Solve this ODE using any method you like

$$u'' + 1u' + 25u = 0 \text{ with } u(0) = 1 \text{ and } u'(0) = 2.$$

(2) (10 pts.) Solve the initial value problem using Laplace transforms

$$\ddot{u} + 2\dot{u} + 101u = 5\sin 10t, \text{ when } u(0) = 0, \dot{u}(0) = 20$$

(3) (20 pts.) Using the series method find a general solution, valid near the origin, of the differential equation

$$8x^2\frac{d^2u}{dx^2} + 6x\frac{du}{dx} + (x-1)u = 0.$$

Where is your solution valid?

(4) (20 pts.) Consider the ode

$$u' \sin x = u \ln u$$

(a) Describe this equation.
(b) Solve this equation by any method you would like.
(c) Solve the initial value problem for the initial condition $u(\pi/3) = e$.
(d) Use mathematica to plot the slope field and your solution. Please include a printout of your solution.
(e) Is the initial value problem well-posed? If so, name the theorem that guarantees this. If not, give two other solutions that are of a different form. Include a printout of these solutions on the slope field.

(5) (15 pts.) Playing with tops on a rainy rain in spring you encounter this set of coupled odes

$$\begin{aligned}\frac{d\omega_1}{dt} &= -\Omega\omega_2 \\ \frac{d\omega_2}{dt} &= \Omega\omega_1 \\ \frac{d\omega_3}{dt} &= 0\end{aligned} \tag{1}$$

where the ω's are angular velocities, and

$$\Omega = \frac{I_3 - I}{I}\omega_3, .$$

I and I_3 are principle moments of inertia and $I_3 > I$. Obtain general solutions for the three components of $\omega^i(t)$. Briefly discuss the motion.

(6) (5 pts.) What is the diagrammatic evaluation of this whale?

(7) (25 pts.) To find the electric (or scalar) potential inside a spherical surface of radius R you can solve the Laplace equation

$$\nabla^2 V = 0.$$

Let's assume that surface is maintained at a constant, non-vanishing potential V_o on $\pi > \theta > \pi/2$ and 0 on $0 < \theta < \pi/2$.

(a) Write down the partial differential equation in terms of spherical coordinates.
(b) What can you say about the φ part of the equation? There is a nice simplification here.
(c) Separate variables and solve the resulting odes.
(d) Write down a general solution for $V(r, \theta)$.
(e) Determine the specific solution for the boundary conditions given above.

(8) (Optional n pts.) Compose a poem on one or more mathematical methods of 320. For instance you might write an ode to your favorite Bessel function.

(9) (20 pts.) **Fraunhoffer Diffraction** In the theory of diffraction of light by a circular aperture, you find that the amplitude A of the diffracted wave depends on the integral

$$A \sim \int_0^a J_o(br)rdr$$

where a is the radius of the circle and $b = 2\pi\sin(\alpha)/\lambda$. The angle α is shown in the figure and λ is the wavelength of the light.

(a) Solve the integral using an appropriate relation.
(b) Intensity is proportional to the square of the amplitude. By looking up the zeros of the Bessel function (found in tables) find the angle of the first dark band (zero intensity) for green light $\lambda = 5.50 \times 10^{-7}$ m. Assume the radius of the circle is 0.500 cm.
(c) Show that the intensity maxima occur at roots of the Bessel function of second order, $J_2(ab) = 0$.

(d) **Extra:** Arfken writes "Had this analysis been known in the 17th century, the arguments against the wave theory of light would have collapsed." Why? [BTW, this is the same analysis that gives the Rayleigh criterion for the resolution of two point sources

$$\Delta\theta = 1.22\frac{\lambda}{d}$$

($d = 2a$).]