General Relativity or "GR" (PHYS 325) Final

Spring 2026 v1.0

Welcome to the GR final! Treat this as a regular problem set *except*

(1) Do not collaborate with others, and
(2) Do not consult resources outside the class materials, which are defined as your own class notes, Schutz, your Mathematica notebooks (and those posted on the class webpage), and Mathematica.
(3) Submit your paper and/or printed solutions by **5 PM** on **May 15**. (Emailed solutions are fine iff they are emailed to me in **one** pdf.)
(4) Regular problem set extensions **do not** apply. (If unforeseen circumstances, like a medical emergency or similar event arise, let me know. We'll work out a schedule for submission.)

Note!

- Problems 1, 2, and 3 are required.
- A perfect score is a grade of 2π. There are 7.29 radians available so you have options and do not need to complete every problem, or every part of every problem.
- Each problem is marked with the number of radians available.

- Enjoy! And may the curvature be with you!

Problems:

(1) **Special Relativity (0.5 rad.)** A railroad boxcar has a clock mounted on its front wall and a clock mounted on its rear wall. Marvin, a robot with a "brain the size of a planet", is at the center of the boxcar. Each clock is attached to a strobe light, and each strobe light is programmed to flash when its local clock strikes midnight. The two clocks are synchronized in the train's frame. The robot Marvin will start an interminable and depressing monologue if it receives a light signal from *one* strobe *or* from the other, but *not* if it gets a signal from *both*. The boxcar travels down a straight track at high speed.

Cixin analyzes this situation from the boxcar's frame, concluding "The two strobes flash simultaneously, the two light signals travel the same distance, and the two light signals reach the package simultaneously. Therefore, Marvin remains silent."

Adams analyzes this situation from the Earth's frame: "The two strobes flash simultaneously, but Marvin is moving away from the rear (left-hand) light signal and toward the front (right-hand) light signal. The front light signal thus has less distance to travel so it makes it to Marvin first. Therefore Marvin begins the monologue."

Clearly, these two accounts cannot both be correct.

(a) Draw space-time diagrams of the events, worldlines, and light signals in both reference frames.

(b) Find the flaw in one analysis, and state definitively whether Marvin speaks or not.

(2) **Tensors (0.5 rad.)** A friend, who knows some math and physics, stops you in the hall and asks, "What is a tensor?" You actually have a bit of time to explain and there is even a handy board in the hall to write on... What would you answer? Give at least two perspectives on what a tensor is, one of which should be short and accurate, and include at least one example.

(3) **Geometry (1.0 rad.)** A $(2+1)$D metric with coordinates (t, θ, φ) is

$$ds^2 = -dt^2 + a(t)^2 \left[d\theta^2 + \sin^2(2\theta) d\varphi^2 \right]$$

where a is a function of time, $0 \leq \theta \leq \pi/2$, and $0 \leq \varphi \leq 2\pi$.

(a) Write the metric $g_{\alpha\beta}$ and inverse metric $g^{\alpha\beta}$ in matrix (or array) form.

(b) Find the circumference of the space at constant time t and θ (so $dt = d\theta = 0$). What is the circumference at $\theta = \pi/4$? What is the maximum circumference?

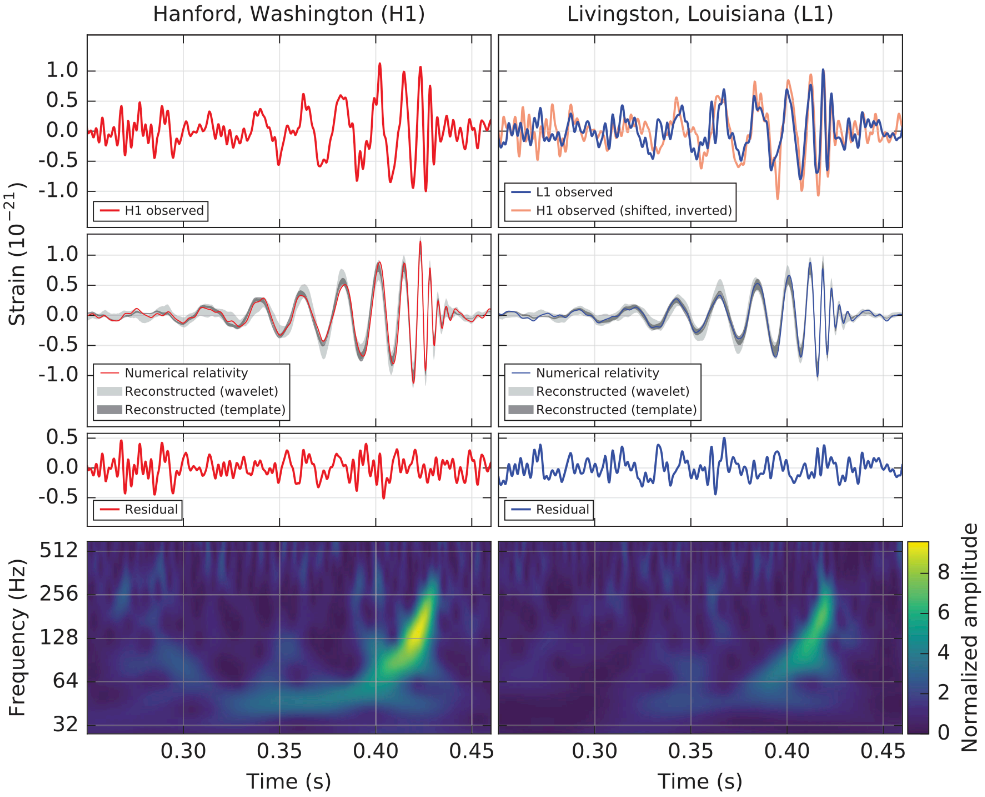
(c) Find the Christoffels for this metric by hand or by Mathematica. Please show your work in your solutions.

(d) By any method find the non-vanishing Riemann components, the Ricci tensor, and the Ricci scalar.

(e) Is the 2D spatial geometry homogeneous and/or isotropic?

(f) On the basis of your findings, please describe and/or sketch this geometry.

(4) **GWs (0.5 rad.)** For the gravitational wave detection GW150914 shown here



Two black holes orbiting each other coalesced into one black hole. The original pair had masses of about 36 $M_\odot$ and 29 $M_\odot$ while the final black has mass 62 $M_\odot$.

(a) Compare the final mass with the initial mass. What is the difference? Where did that mass go?

(b) From the above plots estimate the wave's amplitude, frequency, and rate of change of frequency just before the merger.

(c) Given your estimate of the wave's amplitude and the fact that it decreases as $1/r$ from the source, estimate the distance of the binary black hole system from us.

(d) Find what is called the "chirp mass" $\mathscr{M}$ from

$$\frac{df}{dt} = 12 \left(\frac{\mathscr{M}}{M_\odot}\right)^{5/3} \left(\frac{f}{100\ \text{Hz}}\right)^{11/3} \text{s}^{-2}$$

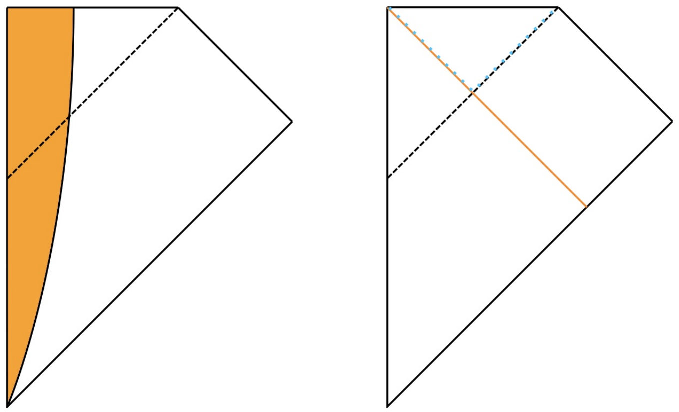
(e) The energy radiated is

$$E = \frac{1}{2}\mathscr{M}\,(\mathscr{M}\omega)^{2/3}$$

where ω is the angular frequency of the wave. What is the radiated energy from your estimates? Is this consistent with what your found before?

Please show all your work.

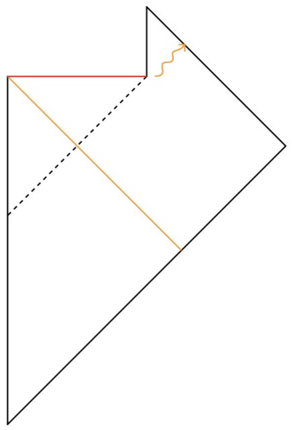
(5) **Penrose diagrams (0.5 rad.)** Here are two, different black hole spacetimes



(a) Fully label one of the diagrams including i_o, i^+, i^-, $\mathscr{I}^+$, $\mathscr{I}^-$, the singularity, and the horizon. If you forget the notation either ask or write brief descriptions of the boundary spaces.

(b) Explain physics represented in both these diagrams in as detailed way as you can by reading the history from bottom (i^-) to top.

(c) What happens in this diagram?



(6) **Schwarzschild red/blueshift (1.0 rad.)** As we saw, the Schwarzschild geometry in Eddington-Finkelstein coordinates is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2. \tag{1}$$

where

$$t = v - r - 2M \ln\left|\frac{r}{2M} - 1\right|. \tag{2}$$

(a) From a great distance away from the black hole drop we drop the effervescently happy robot Colin (as described in Douglas Adam's "Mostly Harmless") from rest into the black hole. On the way down Colin sends cheery periodic updates. We'll use the coordinates (v_e, r_e) for the emission event and (v_r, r_r) (and t_r) for the reception event. Show that our v_r coordinate - far from the black hole -

is

$$v_r \approx t_r + r_r.$$

(b) Find the differential equations for the ingoing and outgoing radial null rays using the metric of equation(1).

(c) Show that the solution for the outgoing (for $r > 2M$) light rays is

$$v = 2\left(r + 2M \ln\left|\frac{r}{2M} - 1\right|\right) + \text{ const.} \tag{3}$$

(d) Just before crossing the horizon Colin's emitted frequency is redshifted enormously with a time dependence (as measured far away) of $e^{-\alpha t_r}$. Find the mass of the black hole in terms of α.

(7) **Cosmology! (1.5 rad.)** (Each part is weighted equally.)

(a) Starting from the FRW metric

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{1}{1-kr^2} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right]$$

with $k = 1$ show that "tt" component of Einstein's equations

$$G_{\alpha\beta} + g_{\alpha\beta}\Lambda = 8\pi T_{\alpha\beta}$$

becomes the FRW equations

$$\dot{a}^2 - \frac{8\pi}{3} a^2 (\rho + \rho_\Lambda) = -k = -1, \tag{4}$$

where $\rho_\Lambda = \Lambda/8\pi$. Please use the energy-momentum tensor

$$(T^\alpha_\beta) \rightarrow \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

or, equivalently,

$$T^{\alpha\beta} = (\rho + p) U^\alpha U^\beta + p\eta^{\alpha\beta}.$$

Feel free to use Mathematica or work by hand. If you use Mathematica please include a printout of your notebook with your solution. Which cosmological model is this?

(b) Suppose that a cosmology had no matter, radiation or curvature ($k = 0$) so only Λ was non-vanishing. Find the solution, $a(t)$, for this "cosmological-constant dominated" universe.

(c) In one alternate model of dark energy called "Q" the cosmological constant Λ is replaced with a self-interacting field. The energy density of this field is $\rho_Q = \rho_{Qo} a^{-\eta}$ for some positive parameter η.Some not very exciting algebra shows that the cosmological constant term $\Omega_\Lambda \tilde{a}^2$ in the effective potential is replaced with $\Omega_Q \tilde{a}^{2-\eta}$.

- (i) Load up the FRW Mathematica notebook and make this modification to the effective potential. Assume a Hubble constant of 74 km/s Mpc^{-1}. Enter currently accepted values for Ω_m and Ω_r, and $\Omega_Q = 0.7$ and $\eta = 0.3$. Run the notebook and interpret the effective potential plot. Verify that interpretation with the plot of $\tilde{a}$ vs $\tilde{t}$.
- (ii) The age of the oldest stars is approximately 12 Gyr. Use the notebook to see if the model allows for the existence of such stars. If not, find values for the Ω's that allow such early stars.
- (iii) What is the ultimate fate of this universe?

Submit a printout of the notebook with your solutions.

(8) **GR (0.5 rad.)** Show that in the cosmological model of equation (4) if the total density, density of cosmological constant, and total pressure always satisfy $\rho + 3p - 2\rho_\Lambda > 0$, such as it would for a dust filled cosmology with cosmological constant, then at some time in the past there was a big bang singularity. Use equation (4) for the scale factor $a(t)$ and

$$\frac{d}{dt}\left(\rho a^3\right) = -p\frac{da^3}{dt}$$

(which is conservation of energy as we saw).

(9) **GR (0.5 rad.)** As a recent paper by J. Fuchs *et. el.* puts it, "Warp drives are exotic solutions of general relativity that offer novel means of transportation." As we have seen, a relatively simple example of Alcubierre's warp drive metric is

$$ds^2 = -dt^2 + [dx - V_s(t)\, f(r_s)\, dt]^2 + dy^2 + dz^2 \tag{5}$$

in which a worldline representing the spaceship, $x = x_s(t)$ has a velocity $V_s(t) = dx_s/dt$. The radius

$$r_s = \left[(x - x_s(t))^2 + y^2 + z^2\right]^{1/2}$$

determines the size of the curved region of spacetime. The dimensionless function defines the "warp bubble" given by

$$f(r_s) = \begin{cases} 1 - \left(\frac{r_s}{R}\right)^4 & \text{for } r_s < R \\ 0 & \text{for } r_s \geq R \end{cases}$$

(a) Discuss this spacetime. What might be the advantage for the ship's company?
(b) Find the timelike normal n_α to a surface of constant t.
(c) Modify the Mathematica notebook to show that

$$T^{\alpha\beta} n_\alpha n_\beta = -\frac{1}{2\pi} V_s^2 (y^2 + z^2) \frac{r_s^4}{R^8}$$

and explain why we might have some trouble building a warp drive.

(10) **GR (0.25 rad.)** Consider solutions to

(From a remarkable early GR text (written in verse!) by Lillian Lieber and illustrated by Hugh Lieber.)

(a) What is the physical characteristic of solutions of this equation, $G_{\sigma\tau} = 0$?

(b) Of the solutions to Einstein's equations we considered this semester, which fall in the category of solutions to this equation?

(11) **GR (0.5 rad.)** As David Morin rhymes

> Greetings! Dear brother from Boulder,
> I hear that you've gotten much older.
> And please tell me why
> My lower left thigh
> Hasn't aged quite as much as my shoulder!

Using the weak field limit of GR, show that gravity slows clocks. In particular show that the elapsed time Δt_h of a clock in a homogenous gravitational field of g at height h above another clock with elapsed time Δt_o is

$$\Delta t_h = \left(1 + \frac{gh}{c^2}\right)\Delta t_o.$$

Explain how this calculation relates to the equivalence principle.