General Relativity (PHYS 325) PS 10

Spring 2024 v1.5

These problems are mostly on the orbits in Schwarzschild spacetime. This week in class we'll explore some methods of understanding spacetimes before moving into gravitational waves.

Reading:

- Schutz Chapter 11 on BH's

Problems:

(1) Make a plot of the effective potential $\mathcal{U}_{eff}$ for massive particles around a Schwarzschild black hole for orbits with $\ell = 4.4M$. Classify the orbits.

(2) An explorer falls radially into a black hole:
   (a) In class we found a conserved quantity e for BH orbits and trajectories. If an infalling explorer starts from rest relative to a stationary observer at $r = 10M$ then show that the value of this conserved quantity e for the observer is $2/\sqrt{5}$.
   (b) How much proper time elapses on this explorer's worldline before hitting the singularity?
   (c) How long would this be for a 10 solar mass black hole?

(3) Consider a spacecraft radially falling into a black hole. Suppose it starts at rest far away so that $e = 1$
   (a) Show that the velocity in Schwarzschild coordinates becomes

$$\frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right)\sqrt{\frac{2M}{r}}$$

   and

$$\frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}}$$

   (b) How much proper time elapses between passing the horizon and arriving at the singularity? Express your answer in terms on M.
   (c) For a $M = 2.6 \times 10^6$ solar mass black hole like at the center of our galaxy, how long is this time?
   (d) Suppose the explorers aboard the spacecraft wish to extend this time. Can they? See if you can guide them including some angular motion in φ, and thus some angular momentum ℓ See if you can find the maximum time for the infall, assuming $e = 1$. It may be helpful to keep in mind that the physical distance around the black hole is given by $r\varphi$ so that the maximum speed is

$$r\frac{d\varphi}{d\tau} = 1$$

(4) New results from quantum gravity: Lewandowski et. al. suggest that the metric for spherically symmetric spacetimes is

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{\beta M^2}{r^4}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{\beta M^2}{r^4}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

where $\beta \simeq \frac{G\hbar}{c^3} \simeq 10^{-66}$ cm^2.

(a) If

$$f(r) = \left(1 - \frac{2M}{r} + \frac{\beta M^2}{r^4}\right)$$

has real roots, what is the minimum mass of such a black hole? Work this out in both the above units and as compared to the sun's mass.

(b) Find the Killing vectors for this metric.

(c) Derive the new geodesic equation for light in the equatorial plane. Sketch or plot the effective potential. Discuss the orbits.