General Relativity (PHYS 325) PS 11

Spring 2024 v1.0

This week we continue work on gravitational waves as ripples on the Minkowski background. What's up is determining the physics in $h_{\alpha\beta}$, finding how to produce this radiation, and how to observe it. The problems are largely on geometry in the metric.

Reading:

- Schutz Chapter 9

Problems:

(1) A spatial metric has the form

$$ds^2 = \frac{dr^2}{(1 - 2R/r)} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right).$$

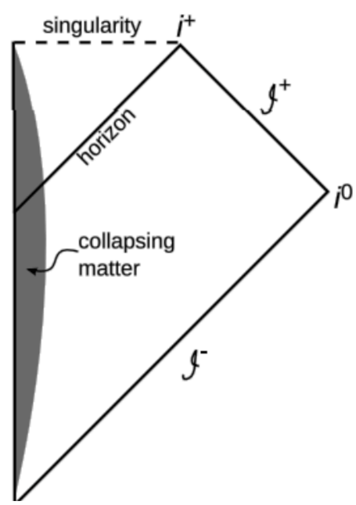
(a) Calculate the radial distance between $r = 2R$ and $r = 3R$. Compare withe usual result in flat space. State your results as multiples of R.
(b) Calculate the spatial volume between spheres at $r = 2R$ and $r = 3R$.
(c) How would this volume compare to the volume between two spherical shells with the same dimensions in flat space? State your results in terms of numbers times πR^3.
(d) Choose the equatorial slice of this 3D space by fixing $\theta = \pi/2$. Now we have a two dimensional metric for a surface with coordinates (r, ϕ). What would this surface look like if you embedded this surface into 3D, as we did in class?

(2) Show that the Einstein's equations in vacuum, $R_{\alpha\beta} = 0$, at first order in $h_{\alpha\beta}$ reduce to

$$-\Box h_{\alpha\beta} + \partial_\alpha V_\beta + \partial_\beta V_\alpha = 0$$

where $V_\alpha = \partial_\gamma h^\gamma_\alpha - \frac{1}{2}\partial_\alpha h$. (We started this in class last Thursday.)

(3) Discuss the main features of this spacetime, as pictured in a Penrose diagram. Compare to the Schwarzschild BH.



(4) 11.8 Gravitational redshift and time dilation around a BH

(5) Alice and Bob hover in rockets outside a Schwarzschild BH of mass M, choosing a position

$$U = \frac{1}{2} = \left(\frac{R}{2M} - 1\right)^{1/2} e^{R/4M}$$

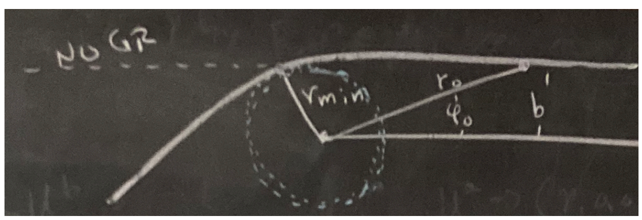
so that $R \simeq 2.16M$. Bob leaves Alice at $t = 0$ and descends into the BH along a straight line in Kruskal-Szerkeres coordinates until arriving at the singularity at $r = 0$, and $U = 0$, $V = 1$. Alice remains at the original location

(a) In a diagram using these U, V coordinates sketch the trajectories of Bob and Alice.

(b) Does Bob follow a timelike wordline?

(c) What is the latest Schwarszchild time after Bob leaves that Alice can send a signal to Bob?

(6) (Optional 1 pt) This problem completes the calculation of the deflection of light around the sun. Here's a sketch:



(a) Starting with our result for the effective potential for light, show that with the change of variables $u = b/r$ the "effective energy conservation" equation for light becomes

$$\left(\frac{du}{d\varphi}\right)^2 = 1 - u^2\left(1 - \frac{2M}{b}u\right).$$

This is for orbits in the equatorial plan.

(b) Using the mass and radius of the sun, $R = 6.96 \times 10^5$ km, show that the quantity $2M/b$ is always small.

(c) Given this, we'll keep only the first order result in M/b for this calculation. Let $y = u(1 - Mu)$ and show that

$$\frac{d\varphi}{dy} \simeq \frac{1 + \frac{2M}{b}y}{\sqrt{1 - y^2}}. \tag{1}$$

(d) The total deflection $\Delta\varphi$ is twice the angle of deflection when integrating from $r \to \infty$ to $r = r_{min}$. We can integrate equation (1) to obtain the angle. Notice that (now) the integration in y separates into two relatively simple integrals. Trace through the change of variables to obtain the limits. Compute these to obtain the result

$$\Delta\phi \simeq \pi + \frac{4M}{b}$$

(e) Assuming the null geodesic just grazes the surface of the sun find the deflection angle $\delta\phi = \Delta\varphi - \pi$.