General Relativity (PHYS 325) PS 2

Spring 2024 v1.7

This second problem set is on special relativity and 4-vectors.

Reading:

Chapter 2 in Schutz.

Problems: Due before class on Thursday Feb 1

All numbered problems are from Schutz.

(1) Roger Penrose, who won the 2020 physics Nobel Prize with Andrea Ghez and Reinhard Genzel, wrote in his 1965 Adam's Prize essay, "The fact that the time interval between two events depends on the route though space-time taken by the idealized clock used to measure it, is often expressed popularly in terms of the 'paradox of the twins': The twin who stays at home experiences a longer time interval than [the twin] who returns home after making a long journey at a speed approaching the speed of light. The twins have thus aged by different amounts when they meet again."

Let's see how this works with an example. A pair of twins try the classic experiment, Aki stays on earth while Bai zips off at $v = 99\%$ the speed of light to Proxima Centauri, a distance of 4.25 lyr from Earth, to check on the planets there. Suppose that (unrealistically) Bai travels at constant speed there and back.

(a) When does Bai return according to Aki?
(b) How much time has elapsed for Bai?
(c) Wait, shouldn't *both* of the twins say the other is younger - "moving clocks run slow", after all? Show that this is not the case since there is a distinguishing event along Bai's worldline.
(d) Draw a spacetime diagram of the history, including surfaces of simultaneity for Bai just before and just after Bai reaches Proxima Centauri. Circle the distinguishing event. Comment on the meaning of the Bai's surfaces of simultaneity for Aki's history.

(2) An experimental test of the "twin paradox" was done in October 1971 with two atomic clocks. J. Hafele and R. Keating took atomic clocks, one eastward, one westward, around the world twice using commercial airlines. When they returned they compared their times with the clock at the US Navel Observatory.

Let's see how this work for a simple case: Using the leading order correction, find the expected disagreement, $\Delta t = t' - t$, between a traveling clock and the clock that stayed "home" on one leg of 14 hours. Please look up speeds of airplanes. (Interestingly, Hafele and Keating had to account for general relativistic effects as well. With these effects, they found agreement between their experiment and the theoretical predictions.)

(3) 1.17 The 'Pole in Barn Paradox' similar to what we discussed in class

(4) A spatial metric has the form

$$ds^2 = dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

(a) Calculate the radial distance (so along a path with $d\theta = 0 = d\varphi$)

$$d = \int ds$$

between $r = 2$ and $r = 10$.

(b) Sketch a little rectangle with edges $d\theta$ and $d\varphi$ on a surface of constant r. Add the line ds to your sketch. What is the infinitesimal of area, dA, on this surface?

(c) Find the area of the surface located at constant $r = R$.

Do your results seem reasonable? In the end these should be familiar results in spherical coordinates.

(5) Cosmic rays are particles that arrive at Earth traveling at extremely high relative velocities ($v \sim 0.998$). Their origin remains something of a mystery. At a height of about 20 km these particles collide with atoms in the upper atmosphere producing a shower of high energy particles including particles called muons. These particles decay rather quickly; as measured in the laboratory a muon at rest has a mean lifetime of 2.197×10^{-6} s. (You may have observed this yourself!) Suppose that a muon is created at a height of 20.0 km moving at $v = 0.998$.

(a) Find the flight time to earth.

(b) Defining

$$f := \frac{\text{mean time of flight}}{\text{mean lifetime}},$$

find f for such muons.

(c) The fraction of muons that survive is given by e^{-f}. How large is this? In this case about 15% survive. Does your number agree?

(d) If not, explain what went wrong with this calculation and correct it.

Remark: If you like, answer this by working in the Earth and muon frames to show that both accounts agree on the percent surviving the flight from the upper atmosphere.

(6) A Federation cruiser is at rest relative to the border of Klingon Space and is in Federation Space. In the Federation cruiser's frame the border is 6 light-hours away. A Klingon battleship zips by the cruiser moving at $\frac{3}{5}c$ towards the border. Let's call this event A and have it designate noon for both frames. A little while later the Klingon battle ship fires a parting shot in the form of a laser beam ("phaser") at the Federation cruiser. The phaser impacts the cruiser at 8 PM, according to the Federation clocks, severely damaging the spaceship.

(a) Sketch a space-time diagram of the history in the cruiser's frame, including the spaceships, border, and phaser.

(b) According to the Federation cruiser's clocks, when does the Klingon ship pass into Klingon Space? Explain how you found this result.

(c) According to the Federation clocks, when did the Klingon battleship fire the phaser? Explain.

(d) Describe the history of events in the Klingon's frame.

(e) Many months later the case comes to Intergalatic Court. The Klingon-Federation Space-time Treaty states that it is illegal for a Klingon (Federation) ship in Federation (Klingon) territory to damage Federation (Klingon) property. The lawyer representing the Klingon ship's captain argues that they are within the letter of the law, since in the ship's frame the damage to the cruiser occurred after the Klingon ship crossed back into Klingon territory. Hence they were not in Federation territory at the time the damage occurred. Did the event of the phaser impacting the cruiser really occur after the Klingon ship crossed into Klingon territory, in their frame?

(f) On the basis of this case, would you recommend that the Intergalatic Congress re-negotiate the treaty to clarify this law? If so, how would you recommend wording the treaty? If not, what is your advice to ship captain's?

(7) 2.1 (a), (c), and (g)

(8) Identify the free and summation indices in $A^\alpha B_{\alpha\beta} = C_\beta$ and $T^{\alpha\mu\nu} A_\mu B_\nu^\lambda = D^{\alpha\lambda}$. (The text calls summation indices "dummy indices".)

(9) **Optional** Write your own rigorous derivation of the Lorentz transformations. Use any method but highlight the necessary assumptions.