General Relativity (PHYS 325) PS 5

Spring 2024 v1.0

In this fifth problem set is devoted to practice with 4-vectors, 1-forms, and metrics.

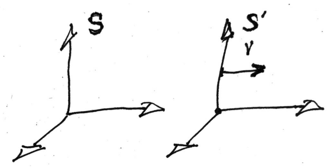
Reading:

Schuta Chapter 3 section 7 - 9 Schutz Chapter 5 sections 2 - 4 (Feel free to read section 5.4 but we will not dwell on non-coordinate bases.)

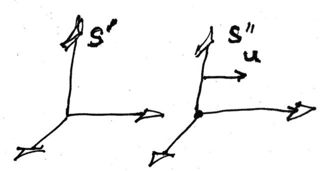
Problems: Solutions due by noon on Friday, February 23

All numbered problems are from Schutz.

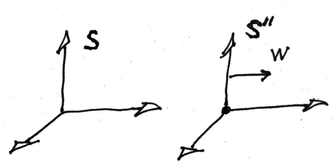
(1) Suppose you have *three* reference frames, $\mathcal{S}$, $\mathcal{S}'$, and $\mathcal{S}$". They all share a common "x" axis. The primed reference frame moves at v along the x-axis in the frame $\mathcal{S}$ as shown in this spatial view:



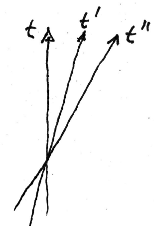
The double-primed reference frame $\mathcal{S}$"moves at u along the x'-axis in the frame $\mathcal{S}'$:



And, of course, the double-primed reference frame $\mathcal{S}$"moves at some speed along the x-axis in the frame $\mathcal{S}$:



Let's call this speed w as shown. Here's a view in spacetrime



What is w in terms of v and u?

(a) Although other ways exist, you can find the answer this way. Let's denote the coordinates of the $\mathcal{S}'$ origin in the frame $\mathcal{S}$ to be $x|_{\mathcal{S}'}$. Let's assume they all synchronize their clocks as they pass. Thus,

$$v = \frac{x|_{\mathcal{S}'}}{t},$$

and

$$w = \frac{x|_{\mathcal{S}''}}{t},$$

By using the inverse Lorentz transformation from $\mathcal{S}'$ to $\mathcal{S}$ (why inverse?) derive the relation between the three speeds.

(b) What is w when $v = u = 0.75$?

(c) What is the maximum value for w?

(d) Check that the result reduces to the Galilean $w = v + u$ for relatively small speeds.

This is a key result of special relativity.

(2) A particle of type "A" crashes into a particle of type "B" (at rest in this frame) producing a shower of particles of type "C", $C_1, C_2, \ldots$. Show that the threshold energy of this process is

$$E_A = \frac{M^2 - m_A^2 - m_B^2}{2m_B}$$

where $M = m_1 + m_2 + \ldots$ is the sum of the masses of the particles of type "C". At threshold, the particles of type "C" go ... *absolutely nowhere*; they are at rest (which makes sense for a threshold, right?).

(3) 3.4 parts (b) and (c)

(4) 3.18 In part (a) only lower the index for vectors A, B, and C. In part (b) raise the index for p, q and r only.

(5) 3.30 parts (f) - (j) Practice with 4-vectors and derivatives. (We did the first half of this problem last week.)

(6) Consider the 2D spacetime metric with coordinates (t, r),

$$ds^2 = -\left(1 - \frac{1}{\sqrt{r^2 + a^2}}\right) dt^2 + \left(1 - \frac{1}{\sqrt{r^2 + a^2}}\right)^{-1} dr^2,$$

where a is a constant.

(a) Write the metric as a 2×2 array in these coordinates.

(b) Find and write the inverse metric as a 2×2 array in these coordinates.

(c) Compute the non-vanishing Christoffel symbols by hand. Please show your work.