General Relativity (PHYS 325) PS 7

Spring 2024 v1.5

This week we extend our skills with the covariant derivative or connection, start computing the Riemann tensor, and practice predictions for particle processes (ah, alliteration!). Now that we have the technology, we'll find the Einstein equations by bridging the gap between his key insight (the equivalence principle) and the mathematics of curvature.

Reading:

Schutz Chapter 6 sections 4 - 6
Coming up soon in class:
Section 5.1
Section 4.2 and 4.4 (You might like the rest of this chapter too!), and
Section 8.1 and 8.2

Problems:

All numbered problems are from Schutz.

(1) Last week you found the Christoffel symbols for the metric

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)dx^i dx_i,$$

where ϕ is some function on spacetime. Assuming ϕ is small compared to 1

$$\begin{aligned}\Gamma^t_{tt} &\simeq \partial_t \phi \\ \Gamma^i_{tt} &\simeq \delta^{ij}\partial_j \phi \\ \Gamma^i_{it} &\simeq \partial_t \phi \\ \Gamma^i_{ii} &\simeq -\partial_i \phi \\ \Gamma^t_{ii} &\simeq -\partial_t \phi \text{ and} \\ \Gamma^t_{it} &\simeq \partial_i \phi\end{aligned} \tag{1}$$

with no sum over i's.

(a) Show that the geodesic equation for 4-velocities implies that

$$p^\alpha \nabla_\alpha p^\beta = 0$$

where p^α is the 4-momentum.

(b) Taking the spatial components of this equation find the equation of motion. By comparing this to $d\mathbf{p}/dt = \mathbf{F}$ explain what ϕ is.

(c) By taking the time-like component of the above equation, show that the particle's energy is conserved if ϕ is time-independent.

(d) (Let's do this one together in class on Thursday) Find R^t_{iit} (no sum over i here) in the limit of small ϕ.

(2) 5.15 Practice with taking derivatives. Don't recompute the Christoffels for polar coordinates. Instead use your notes or Eq. (5.45).

(3) 6.13 A key result for geodesics

(4) For the metric

$$(g_{\alpha\beta}) \to \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \quad (2)$$

some of the non-vanishing Christoffel symbols are

$$\Gamma^t_{rt} = \frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}, \Gamma^r_{tt} = \frac{M}{r^2}\left(1-\frac{2M}{r}\right), \Gamma^r_{rr} = -\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}, \Gamma^t_{tt} = 0$$

(a) Find the radial acceleration $a^r = U^\alpha \nabla_\alpha U^r$ for a stationary observer at a radius r with 4-velocity U^α.

(b) Find the magnitude of the acceleration for this observer.

(5) One channel for pion production is

$$\gamma + p \to n + \pi^+$$

(or, "a proton collides with a photon producing a neutron and a pion".)

(a) Find the minimum or 'threshold' energy the photon would have to have to produce a pion in this channel.

(b) In the Large Hadron Collider (LHC) at CERN protons reach an energy of 6.5 TeV. What is the γ factor for these protons? Can you walk or run as fast as the difference between the protons' speed and c?

(c) Is this reaction within reach of the LHC?

Handy masses include $m_\pi \simeq 140$ MeV/c^2 and $m_p \simeq m_n \simeq 940$ MeV/c^2. And 1 TeV is 10^6 MeV.