General Relativity (PHYS 325) PS 8

Spring 2024 v1.5

Before break we motivated the Einstein's equations from the ideas of curved spacetimes, the equivalence principle, energy conservation, and the Newtonian limit. Now we can solve for a bunch of different, physically relevant cases. We've started with the Schwarzschild black hole - working out way down to a simplified form of the metric based on spherical symmetry. This week we'll finish the black hole solution sand begin our study of these fascinating things.

Reading:

- Schutz Chapter 10 sections 1 - 5
- Before break we discussed Schutz's section 5.1, sections 8.1 and 8.2, and Chapter 7

Problems:

All numbered problems are from Schutz.

(1) Compute the Schwarschild radius of a 30 solar mass black hole in geometrized units and km. This is roughly the size of the black holes involved in the first gravitational wave detection.

(2) (worth 4 points) The beginning of the analysis of curvature quantities for "our metric"

$$(g_{\alpha\beta}) \to \begin{pmatrix} -e^{2\phi} & 0 & 0 & 0 \\ 0 & e^{2\lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{1}$$

Note that $\phi = \phi(r)$ is a function or r in the metric and φ is the coordinate. Both ϕ and λ *are functions of* r *only.* Let's denote the derivatives with primes so that, for instance,

$$\frac{d\phi}{dr} = \phi'$$

In this problem please compute at least one component of the Christoffels and the Riemann tensor by hand, then compute the rest with the Mathematica notebook.

(a) Find the inverse metric and then the Christoffel symbols. So you can check your work here's a couple Christoffels

$$\Gamma^t_{tr} = \phi' \text{ and } \Gamma^r_{\theta\theta} = -re^{-2\lambda}$$

(b) Now compute the components of Riemann. Here's a couple of components to get you started,

$$R^t_{rtr} = -\phi'' + \lambda'\phi' - (\phi')^2 \text{ and } R^\theta_{\varphi\theta\varphi} = \sin^2\theta\left(1 - e^{-2\lambda}\right)$$

(c) Find the non-vanishing components of the Ricci tensor $R_{\alpha\beta}$, the Ricci scalar R, and finally the Einstein tensor $G_{\alpha\beta}$.

(3) (2 pts.) One of the best ways to identify spacetime symmetries and the associated conserved quantities is through Killing vectors (so named for Wilhelm Killing, a German mathematician) and one of the easiest ways of identifying these vectors is by asking - does the metric depend on each coordinate? For example,*if the metric doesn't depend on the coordinate* x^3 then the transformation

$$x^3 \to x'^3 = x^3 + \text{ any constant}$$

leaves the metric unchanged. The vector

$$\xi^\alpha \to (0, 0, 0, 1)$$

is in the direction where the metric doesn't change. As you can see,

$$x'^\alpha = x^\alpha + \epsilon\xi^\alpha \text{ or } x'^3 = x^3 + \epsilon$$

for an infinitesimal shift by ϵ. This vector ξ is a Killing vector for the translational symmetry of x^3.

(a) Find the 3 similar translational symmetries of the Minkowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

There are 6 more symmetries. Can you identify them?

(b) Killing vectors satisfy the equation

$$\nabla_{(\alpha}\xi_{\beta)} = \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha = 0.$$

To see why it is useful, show that if the vector field ξ satisfies the above equation then the quantity $p^\alpha\xi_\alpha$ is constant along a geodesic. This is a conserved quantity! Since this holds for each Killing vector, there is an associated conserved quantity for each symmetry.

(c) An infinitesimal Lorentz transformation with speed v in the x direction can be written as

$$\begin{aligned} x'^0 &= x^0 - vx^1 \\ x'^1 &= x^1 - vx^0 \end{aligned} \tag{2}$$

Show that the Killing vector $\xi^\alpha \to (x, t, 0, 0)$ corresponds to this infinitesimal transformation. Sketch ξ^α in a few places in the $t - x$ plane, in the quadrant $x > 0$ and $x > |t|$.

(d) Show that this Killing vector satisfies Killing's equation.