# Stat Mech (PHYS 370) Guide 10 Solutions

Fall 2024 v2.0

**Problems:**

(1) For the relativistic case: When the momentum is high we have

$$E = \sqrt{(pc)^2 + m^2c^4} = (pc)\sqrt{1 + \frac{m^2c^2}{p^2}} \simeq pc + \frac{1}{2}\frac{m^2c^3}{p}$$

where I have used the beginning of the binominal theorem, $(1+x)^n \simeq 1 + nx$, which works well when $x$ is small. In this case if $mc/p < 1$ then the approximation works well.

(2) The probability of occupation is given by the Fermi-Dirac distribution so with $kT \simeq 1/40$ eV at room temperature we have

(a) At this difference,

$$P_a = \bar{n}_{FD} = \frac{1}{e^{\beta(\epsilon-\mu)} + 1} = \frac{1}{e^{-40} + 1} \simeq 1$$

(b) At this difference,

$$P_b = \frac{1}{e^{\beta(\epsilon-\mu)} + 1} = \frac{1}{e^{-.4} + 1} \simeq 0.6$$

(c) Well, this is 1/2 as we saw in class:

$$\bar{n}_{FD}(\mu = \epsilon) = \frac{1}{e^0 + 1} = \frac{1}{2}$$

(d) At this difference,

$$P_d = \frac{1}{e^{\beta(\epsilon-\mu)} + 1} = \frac{1}{e^{0.4} + 1} \simeq 0.4$$

(e) At this difference,

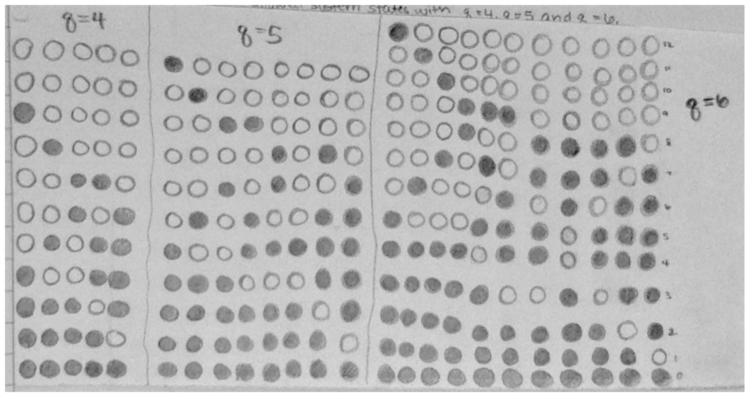
$$P_e = \frac{1}{e^{\beta(\epsilon-\mu)} + 1} = \frac{1}{e^{40} + 1} \simeq 0$$

These differences carry us through the transition from occupied states to unoccupied states.

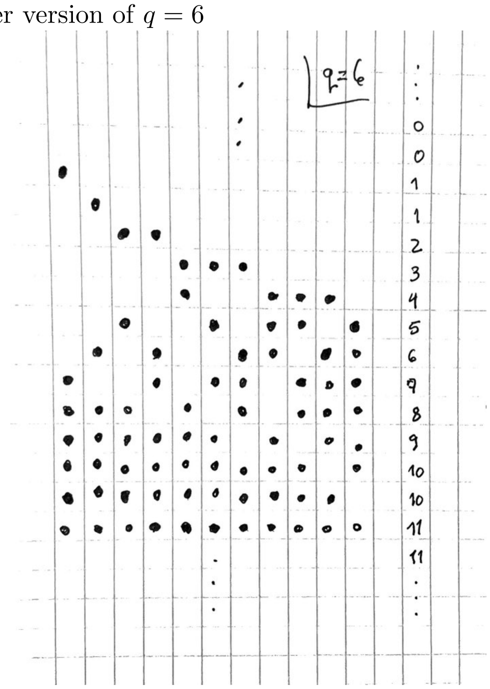
(3) (3 pts.) A numerical computation of $\mu(T)$.

(a) The dot diagrams. Some of these are easy - the ones on the left sides below. The others can be very tricky. The idea is that you want to partition the total number of energy units among the stack of fermion states. For instance the states on the far left raise the topmost fermion $q$ units while the one on the far right raise the top $q$ fermions one unit. Here they are[1]

[1] Thanks to Patricia for the diagram!



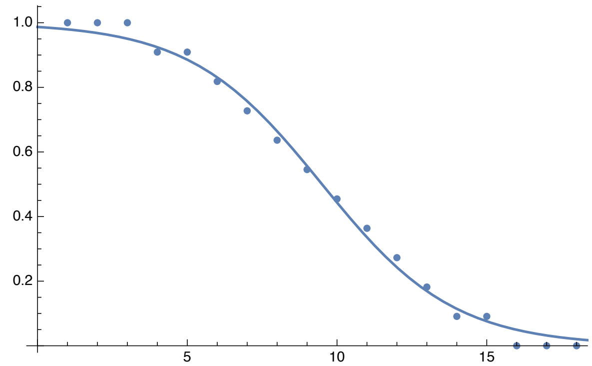

There are many empty states above these and many full states below these. The 6th column in $q = 5$ is a typo. There are only 7 of these states. There are typos in the $q = 6$ case too. Here's another version of $q = 6$




(b) For $q = 6$ we count the rows in the diagram, and order them in ascending order, to obtain

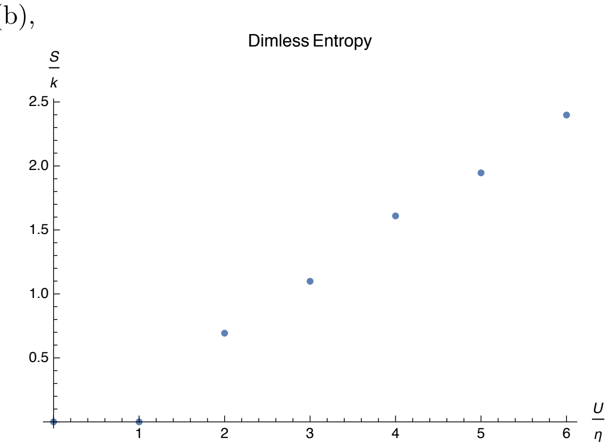
$$\ldots, \frac{11}{11}, \frac{11}{11}, \frac{11}{11}, \frac{10}{11}, \frac{10}{11}, \frac{9}{11}, \frac{8}{11}, \frac{7}{11}, \frac{6}{11}, \frac{5}{11}, \frac{4}{11}, \frac{3}{11}, \frac{2}{11}, \frac{1}{11}, \frac{1}{11}, \frac{0}{11}, \frac{0}{11}, \frac{0}{11}, \ldots$$

where I added the empty and full states that I asked for in the problem set. The plot with the fit looks like this




(c) The chemical potential is determined by $\overline{n} = 1/2$, which occurs in the plot at about 9.5 or $9.5\epsilon$, including the energy scaling. From the fit the inverse dimless temperature is 0.456 so $kT = 2.19\epsilon$. (Mathematica will also give you uncertainties but let's save that for another day.)

(d) So the entropy is $k \ln \Omega$ so we can easily make the plot by taking the logs of the multiplicities in parts (a) and (b),




Hmm, pretty linear! The difference between ln(11) and ln(7) is 0.452. Inverting this gives 2.21 so $kT = 2.21\epsilon$ which agrees with the previous result to two sig figs.

(4) Conduction electrons in copper. First, to find the volume of one mole from the molar mass and the density I found

$$V_1 = \frac{63.5 \text{ g}}{8.93 \text{ g/cm}^3} \simeq 7.11 \times 10^{-6} \text{ m}^3.$$

For an Avagadro's number of electrons with mass $m_e = 9.11 \times 10^{-31}$ kg, the Fermi energy is

$$\epsilon_F = \left(\frac{h^2}{8m_e}\right)\left(\frac{3N}{\pi V}\right)^{2/3} \simeq 1.1 \times 10^{-18} \text{ J} \simeq 7.05 \text{ eV}.$$

The Fermi temperature is just $\epsilon_F/k$ or

$$T_F \simeq 8.2 \times 10^4 \text{ K}$$

That's high! Certainly room temperature is approximately "0" compared to the Fermi temperature; in light of the last problem certainly $T_F \gg T$. The electron gas is degenerate.

From the text on page 275 the degeneracy pressure is

$$P = \frac{2}{5}\frac{N}{V}\epsilon_F \simeq 3.8 \times 10^{10} \text{ Pa} = 3.8 \times 10^{5} \text{ atm}$$

while the bulk modulus is

$$B = \frac{10}{9}\frac{U}{V} = \frac{5}{3}P \simeq 6.4 \times 10^{10} \text{ Pa} = 6.4 \times 10^{5} \text{ atm.}$$

This is about half of the quoted bulk modulus of $1.2 \times 10^{11}$ Pa !