Stat Mech (PHYS 370) Guide 11 Solutions

Fall 2024 v1.0

Problems:

(1) New bosonic partition function?

(a) For this example we had $Z_1 = 4$ and so

$$Z_b = \frac{Z_1^N - Z_1}{N!} + Z_1 = \frac{4^2 - 4}{2!} + 4 = \frac{12}{2} + 4 = 10$$

which is indeed what we found.

(b) I will grade parts (b) and (c)

(2) This looks like a good Mathematica question. From the file appended at the end of this solution the distributions are all at least within 1% when $x = \epsilon - \mu \simeq 0.133$ eV.

In the atmosphere... Hmmm, there's a lot in play here. To get an estimate let's work with an "average Earth atmosphere" particle. This will have $m = \bar{m}_{\text{air}} \simeq 28.96$ g/mol from our solution to 1.14 and have the same rotational partition function as N_2. For such a gas,

$$\mu = kT \ln\left(\frac{N\ell_Q^3}{Z_{int}V}\right) = kT \ln\left(\frac{P\ell_Q^3}{kTZ_{rot}}\right)$$

using the ideal gas law. $Z_{rot} = KT/2\epsilon_R$ where $\epsilon_R = 2.5 \times 10^{-4}$ eV (see pg 236 and Quiz II). On the surface (at 1 atm) at 300 K,

$$\ell_Q = \simeq 1.9 \times 10^{-11} \text{ m.}$$

and

$$\mu \simeq -0.5$$

Since $x = \epsilon - \mu$ and the minimum value of ϵ is 0, this is safely above the limit of 0.133 eV under these conditions.

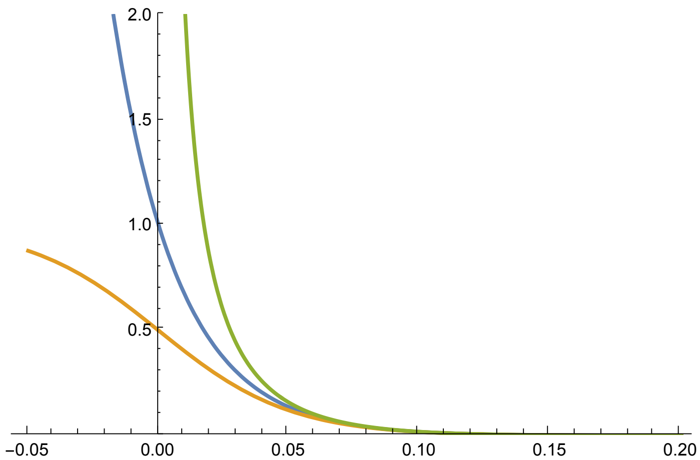
Now using $T = 100$ K gives $\ell_Q = \simeq 3.2 \times 10^{-11}$ m and $\mu \simeq -0.135$ eV so now we would be getting close to the limit. Interesting! I guess planetary atmospheres might be affected by quantum statistics on colder planets.

Here's the notebook file

```
(* computing differences of disributions let x =
 ϵ - μ. Also recall that at room temperature kT = 1/40 eV *)
```

In[78]:=
```
Plot[{Exp[-x * 40], (Exp[x * 40] + 1) ^ (-1), (Exp[x * 40] - 1) ^ (-1)},
 {x, -.05, .2}, PlotRange → {0, 2}]
```

Out[78]=




```
(* Since the Boltzmann distribution is between the FD
 (the lower curve) and the BE (th eupper curve) dsitributions,
then it will be within 1% if they are. So let's look at FD-BE/FD  *)
```

In[82]:=
```
FindRoot[
 ((Exp[x * 40] - 1) ^ (-1) - (Exp[x * 40] + 1) ^ (-1)) / (Exp[x * 40] + 1) ^ (-1) - .01, {x, 0.1}]
```

Out[82]=

```
{x → 0.132583}
```

```
(* normalizing with Boltzmann doesn't make a difference ... *)
```

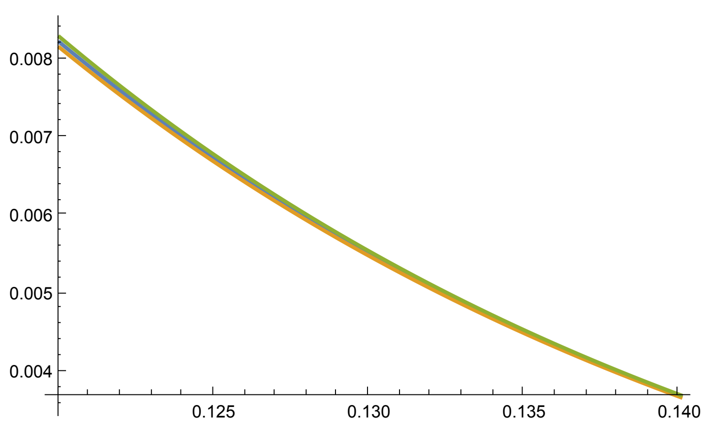
In[71]:=
```
FindRoot[((Exp[x * 40] - 1) ^ (-1) - (Exp[x * 40] + 1) ^ (-1)) / Exp[-x * 40] - .01, {x, 0.2}]
```

Out[71]=

```
{x → 0.132459}
```

In[84]:=
```
Plot[{Exp[-x * 40], (Exp[x * 40] + 1) ^ (-1), (Exp[x * 40] - 1) ^ (-1)}, {x, 0.12, .14}]
```

Out[84]=



In[85]:=
```
(* Yup, all pretty close in this range *)
```

In[110]:=
```
(* Now working on the atmosphere part *)
```

In[105]:=
```
T = 300
```

Out[105]=
```
300
```

In[87]:=
```
(* quantum length *)
```

In[106]:=
```
lQ = N[6.63 * 10 ^ (-34) / Sqrt[2 * Pi * 29 * 1.66 * 10 ^ (-27) * 1.381 * 10 ^ (-23) * T]]
```

Out[106]=

1.87289×10^{-11}

In[89]:=
```
(* in eV *)
```

In[95]:=
```
mu = 8.62 * 10 ^ (-5) * T * Log[
    2 * 2.5 * 10 ^ (-4) * 1.0 * 10 ^ 5 / (8.62 * 10 ^ (-5) * T) (1 / (1.381 * 10 ^ (-23) * T) * lQ ^ 3)]
```

Out[95]=
```
-0.506931
```

```
(* - μ is greater than 0.13 ! *)
```

In[107]:=
```
T = 100
```

Out[107]=
```
100
```

In[108]:=
```
lQ = N[6.63 * 10 ^ (-34) / Sqrt[2 * Pi * 29 * 1.66 * 10 ^ (-27) * 1.381 * 10 ^ (-23) * T]]
```

Out[108]=

3.24395×10^{-11}

In[109]:=
```
mu = 8.62 * 10 ^ (-5) * T * Log[
    2 * 2.5 * 10 ^ (-4) * 1.0 * 10 ^ 5 / (8.62 * 10 ^ (-5) * T) (1 / (1.381 * 10 ^ (-23) * T) * lQ ^ 3)]
```

Out[109]=
```
-0.135832
```

```
(* Oh, that is close! *)
```

(3) (4 pts.) Obtaining the mass limit of white dwarf stars assuming *constant* density. The solution is more or less as we did in class.

(a) You can find the dependence on G, M, and R through dimensional analysis but since we found the exact result in class, let's do this. With constant density,

$$R^3 = \frac{3M}{4\pi\rho},$$

and so the gravitational potential energy of a constant density sphere is

$$\begin{aligned} U_g &= -\int \frac{GM}{R} dM = -G\left(\frac{4\pi\rho}{3}\right)^{1/3} \int_0^M M^{2/3} dM \\ &= -G\left(\frac{4\pi\rho}{3}\right)^{1/3} \frac{3}{5} M^{5/3} \\ &= -\frac{3}{5}\frac{GM^2}{R} =: -\frac{\beta}{R}. \end{aligned}$$

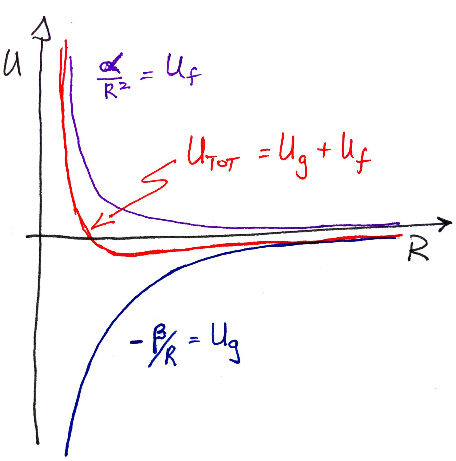
You can also use the constant density to eliminate M and integrate with respect to R, with the same result. I have defined a quantity "β" (Ack! not to be confused with the inverse temperature).

(b) The relation between N (of electrons) and the mass of the star M is $M \simeq 2Nm_p$. This factor of 2 depends on the composition of the star. We assume one electron is paired with 1 neutron and 1 proton, on average, which works well for white dwarf stars that are mostly carbon. Computing the kinetic energy of the degenerate electron gas,

$$\begin{aligned} U_f &= \frac{3}{5} N\epsilon_F = \frac{3h^2}{40m_e}\left(\frac{M}{2m_p}\right)\left(\frac{9M}{8\pi^2 m_p R^3}\right)^{2/3} \\ &= \left(\frac{3h^2}{40m_e}\right)\left(\frac{9}{4\pi^2}\right)^{2/3}\left(\frac{M}{2m_p}\right)^{5/3}\frac{1}{R^2} \\ &=: \frac{\alpha}{R^2} \end{aligned} \quad (1)$$

where I have defined a constant α.

(c) The total energy $U_{tot} = U_f + U_g$ looks something like this



We can obtain the equilibrium radius R_* by differentiation

$$\frac{dU}{dR} = 0 \implies \frac{\beta}{R_*^2} - 2\frac{\alpha}{R_*^3} = 0 \text{ or } R_* = \frac{2\alpha}{\beta}$$

With the constants inserted this is approximately

$$R_* \simeq 0.03 \frac{h^2}{G m_e m_p^{5/3} M^{1/3}} \tag{2}$$

which decreases with increasing mass.

(d) For a one solar mass star, $R_* \simeq 7200$ km. The (constant) density is

$$\rho_* = \frac{M}{\frac{4\pi}{3} R_*^3} \simeq 1.3 \times 10^9 \text{ kg/m}^3$$

which is about a million times larger than the density of water.

(e) With $V = 4\pi R^3/3$ and $N = M/(2m_P)$ the Fermi energy works out to be

$$\epsilon_F = \left(\frac{h^2}{8m_e}\right)\left(\frac{3N}{\pi V}\right)^{2/3} = \left(\frac{h^2}{8m_e}\right)\left(\frac{9M}{8\pi^2 m_p R^3}\right)^{2/3} \simeq 1.9 \times 10^5 \text{ eV} = 0.19 \text{ MeV} = \epsilon_\odot$$

While the Fermi temperature is $T_F = \epsilon_F / k \simeq 2.3 \times 10^9$ K. White dwarf stars are actually at $\sim 10^4 - 10^5$ K so the thermal energy is lower than this and thus we were fine neglecting this thermal energy in the computation. (!)

(f) Now the delicate part. How did we determine the onset of instability? When the star goes relativistic! The energy for an (ultra-)relativistic degenerate electron is

$$E \simeq pc = \frac{hnc}{2L}$$

by the quantum $p \to hn/2L$. (You might wonder how it is that we can use the same quantization for momenta - the Schrödinger equation is not relativistic - but the quantization of momenta is preserved.) Thus the total energy is

$$\begin{aligned} U_{f\,rel} = \langle E \rangle &= 2\int_0^{\pi/2} d\varphi \int_0^{\pi/2} \sin\theta d\theta \int_0^{n_{max}} n^2 \left(\frac{hnc}{2L}\right) dn \\ &= \frac{3}{4} N \epsilon_F = \frac{3}{8}\left(\frac{M}{2m_p}\right)^{4/3}\left(\frac{3}{\frac{4}{3}\pi^2 R^3}\right)^{1/3} \\ &=: \frac{\alpha_{rel}}{R} \end{aligned} \tag{3}$$

Ah so! Now the energy of the electron gas scales with $1/R$ - *which is the same radial scaling as the gravitational energy*. Without the $1/R^2$ of the non-relativistic case, the star cannot be stable and will either expand or contract. To understand when this occurs we need to see when the electrons are relativistic. This occurs when the average kinetic energy is about the same as the rest energy,

$$\bar{\epsilon} = 0.6\epsilon_F \simeq m_e c^2 \simeq 0.511 \text{ MeV}.$$

If we look back at equations (3) and (1), we see that U_f scales with mass to some power ($M^{4/3}$ and $M^{5/3}$, respectively). Thus the electrons become increasingly relativistic with increasing mass. The average electron energy scales with ϵ_F so using "$\sim$" to indicate how the quantity scales with mass,

$$\epsilon_F \sim \left(\frac{N}{V}\right)^{2/3} \sim \frac{M^{2/3}}{R_*^2}$$

where in the second step I have used the linear relation between particle number and mass and have used $V \sim R^3$. But from equation (2) the equilibrium radius depends on the mass of the star through $M^{-1/3}$. So

$$\epsilon_F \sim \frac{M^{2/3}}{R_*^2} \sim M^{4/3}$$

and the average energy also scales with mass to the 4/3. With this mass scaling we can use the ratio of the average energies in the relativistic and solar mass cases to derive up a mass limit. The system will go relativistic when

$$\frac{\bar{\epsilon}_{rel}}{\bar{\epsilon}_\odot} = \frac{0.511}{(0.6)(0.19)} \simeq 4.48 = \left(\frac{M_{rel}}{M_\odot}\right)^{4/3}$$

which becomes

$$M_{rel} \simeq 3.1 M_\odot,$$

or about 5.8×10^{30} kg. Incidentally the actual white dwarf star mass limit - called the Chandrasekhar mass limit - is about 1.4 solar masses. The difference comes from the assumption of a constant density and neglecting relativistic effects.
For fun, I have posted a paper on the course website about computing this mass including possible quantum gravity corrections to special relativity.

(4) These integrals,

$$I_n = \int_{-\infty}^{\infty} \frac{x^n e^x}{(e^x+1)^2} dx$$

are $I_n = 1, 0, \pi^2/3$ for $n = 0, 1, 2,$ and 3, respectively. Here's a clip from a notebook:

```
Integrate[x^2*Exp[x]/(Exp[x]+1)^2, {x, -Infinity, Infinity}]
```

$$\frac{\pi^2}{3}$$

You can read more about these integrals in Appendix B.

(5) Modeling the system as a gas:
(a) Starting with the Fermi energy and keeping in mind for the given molar volume $N/V = N_A/37\ \text{cm}^3$,

$$\epsilon_F = \left(\frac{h^2}{8m}\right)\left(\frac{3N}{\pi V}\right)^{2/3} \simeq 4.3 \times 10^{-4}\ \text{eV}$$

which yields a Fermi temperature of $T_F = \epsilon_F/k \simeq 5.0$ K.
(b) At low temperatures the heat capacity is given by (7.48),

$$C_V = \frac{\pi N k T}{2T_F} \simeq (1.0\ \text{K}^{-1}) NkT,$$

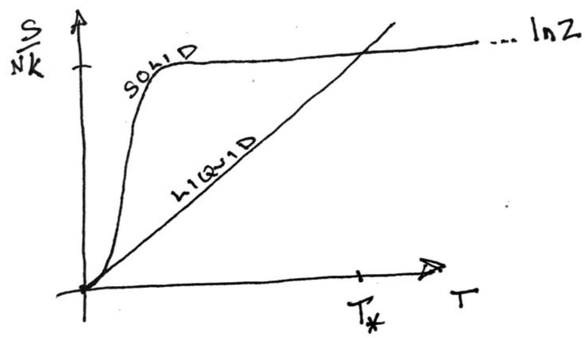
which has the correct linear dependence but is of by 2.8 relative to the experimental data. (Not bad for a gas approximation!)
(c) Now using the experimental result to compute the entropy of the liquid

$$S_l = \int_0^T \frac{C_V}{T} dT = 2.8NkT \implies S/NK = 2.8T.$$

Meanwhile for a solid at low temperatures, just above $T = 0$ ($S = 0$ at $T = 0$), the entropy $S_s = k \ln \Omega_s$ should be

$$\frac{S_s}{Nk} = \ln 2,$$

since ^{3}He has a degeneracy of 2 at low temperatures and so $\Omega_s = 2^N$. Here's the sketch of these two entropies



I've added a guess for the solid curve from absolute zero to just above. The temperature T_* is determined by the intersection

$$2.8T_* = \ln 2 \implies T_* = \frac{1}{4}$$

which is not far form the experimental value of 0.3 given in figure 5.13.

With the Clausius-Clapeyron relation,

$$\frac{dP}{dT} = \frac{S_l - S_s}{\Delta V}$$

this also shows why the phase boundary curve has negative slope below T_* and positive slope above T_*, since $S_l - S_s$ is negative below and positive above T_*.