Stat Mech (PHYS 370) Guide 12 Solutions

Fall 2024 v1.0

Problems:

(1) Setting the derivative of the Planck spectrum to 0 gives

$$3(e^x - 1) - xe^x = 0$$

where $x = \epsilon\beta$. This is a transcendental equation which may be solved with mathematica's `FindRoot` command. The result is approximately 2.82.

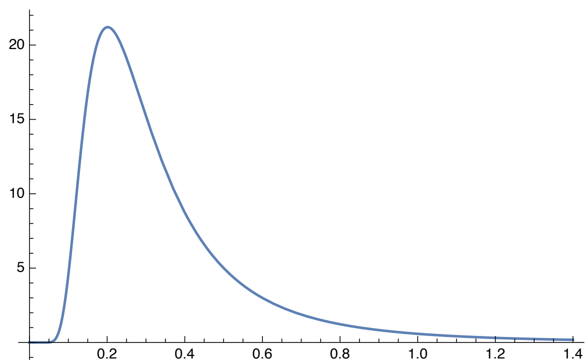
(2) The wavelength is $\lambda = hc/\epsilon$ which means that

$$d\epsilon = -\frac{hc}{\lambda^2}d\lambda$$

So that the energy density becomes

$$\frac{U}{V} = 8\pi hc\int_0^\infty \frac{1}{\lambda^5}\frac{1}{e^{hc\beta/\lambda} - 1}d\lambda.$$

The spectrum is the integrand which looks like



in terms of a dimensionless wavelength $\lambda kT/hc$. Clearly the peak is a wee bit above 0.2. Using `FindRoot` I find that it is at 0.2014 so

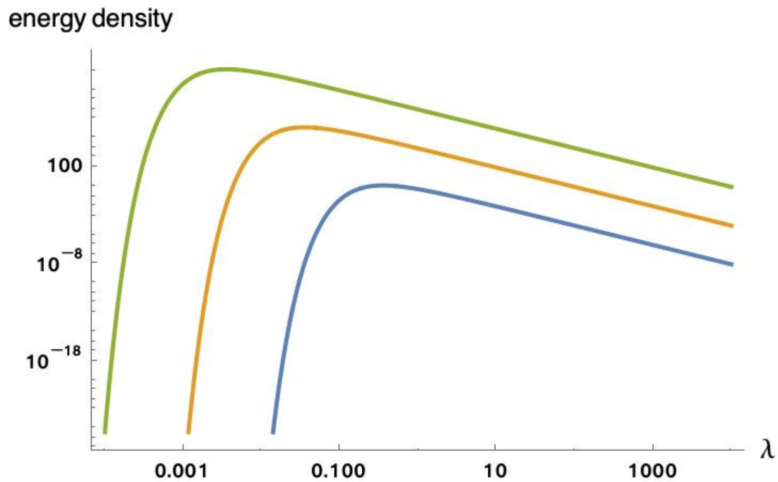
$$\lambda_{peak} = (.2014)\frac{hc}{kT} = \frac{hc}{4.97kT}.$$

This is not the same as what one would think from the peak energy. This is because the relationship between energy and wavelength is non-linear.

(3) The Planck curve is "the intensity versus the product of the wavelength and temperature". Well, the intensity is proportional to the energy density so the spectrum should be proportional to

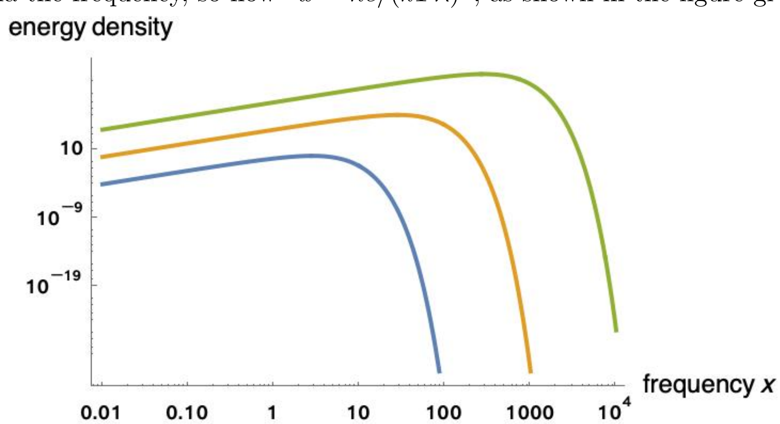
$$I \propto u(\epsilon) \propto \frac{\epsilon^3}{e^{\beta\epsilon} - 1} = \left(\frac{hc}{\lambda}\right)^3 \frac{1}{e^{hc/(kT\lambda)} - 1}$$

where in the second equality I expressed the energy ϵ in terms of wavelength $\epsilon = hc/\lambda$. The exponent contains "the product of the wavelength and temperature". So we have a $T\lambda$ and we can plot the spectrum vs. $x = kT\lambda/(hc)$ giving,



I have rescaled x a couple of times - thus changing T from 1 K (blue) to 10 K (organge) to 100 K (green) - to show that the shape is invariant.

Plotting via the frequency, so now "$x = hc/(kT\lambda)$", as shown in the figure gives



Again, the shape doesn't change as the temperature changes.

(4) Properties of sunlight

(a) At the sun's surface we have an energy density of

$$\frac{U}{V} = \frac{8\pi^5 (kT)^4}{15(hc)^3} \simeq 0.855 \text{ J/m}^3$$

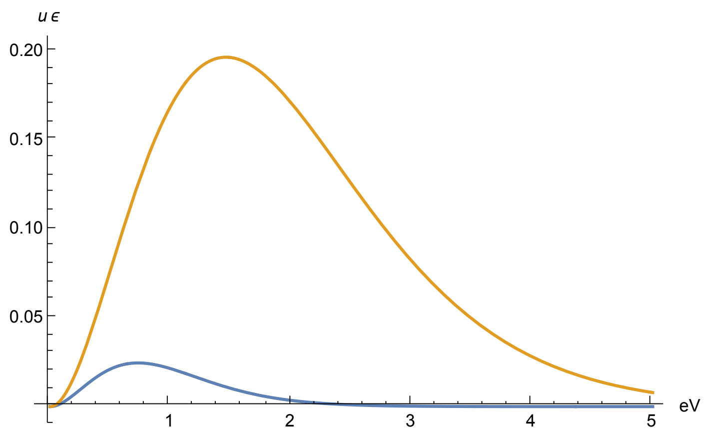
so one cubic meter will contain 0.855 J.

(b) The mathematica notebook is (with a bonus of a extra plot)

In[∘]:=
```
(* 7.38 Plotting the spectra at the two temperatures,
the horizonal axis is in eV *)
```

In[∘]:=
```
Plot[{ϵ^3 / (Exp[ϵ / (3000 * 8.62 * 10^(-5))] - 1),
  ϵ^3 / (Exp[ϵ / (6000 * 8.62 * 10^(-5))] - 1)},
 {ϵ, 0, 5}, AxesLabel → {HoldForm[eV], HoldForm[u (ϵ)]}]
```

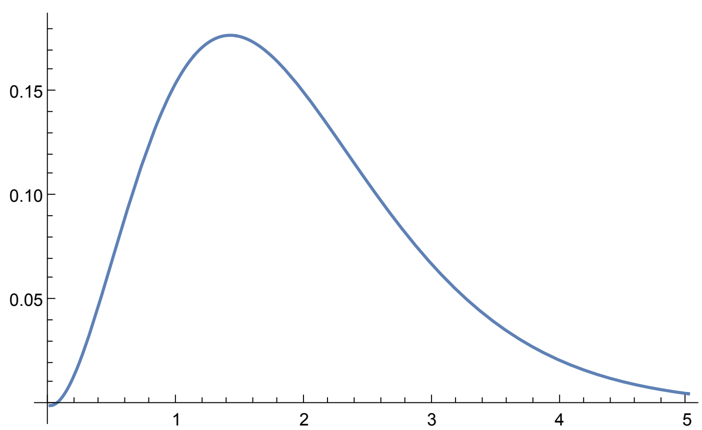
Out[∘]=



In[∘]:=
```
(* 7.43 and now on the visible part ...*)
```

In[∘]:=
```
Plot[ϵ^3 / (Exp[ϵ / (5800 * 8.62 * 10^(-5))] - 1), {ϵ, 0, 5}]
```

Out[∘]=



In[∘]:=
```
(* finding the area in the visible *)
```

In[∘]:=
```
NIntegrate[ϵ^3 / (Exp[ϵ / (5800 * 8.62 * 10^(-5))] - 1), {ϵ, 1.77, 3.1}]
```

Out[∘]=
```
0.14943
```

In[∘]:=
```
NIntegrate[ϵ^3 / (Exp[ϵ / (5800 * 8.62 * 10^(-5))] - 1), {ϵ, 0, Infinity}]
```

Out[∘]=
```
0.405741
```

In[∘]:=
```
N[.149429962601130 / 0.405741]
```

Out[∘]=
```
0.368289
```

In[ ]:=
```
(* That's it! About 37% of sunlight is in the visible *)
```

(c) As computed in the file, about 37% of the light is in the visible, between 1.8 eV and 3.1 eV.

(5) Black holes!

(a) For the 'typical' wavelength I'll just use the peak wavelength that we just found in 7.39.

$$\lambda_{peak} = (0.2014)\frac{hc}{kT_{BH}} = \frac{(0.2014)(hc)}{k} \cdot \frac{16\pi^2 GM}{hc^3} \simeq 15.9\frac{2GM}{c^2}.$$

The last fraction is the Schwarschild radius of the horizon, about 3 km for a solar mass black hole. So the peak wavelength is about 47 km, about 16 times the radius (!). This wavelength is what would be measured far from the black hole. Due to gravitational redshift the wavelength near the black hole would be much shorter.

(b) The power radiating from the black hole is

$$P = \sigma A T^4 \simeq 9 \times 10^{-31} \text{ W}.$$

This is really small! The black hole might radiate a low energy photon every few seconds.

(c) The power in radiation can only come from the mass so

$$\frac{d\left(Mc^2\right)}{dt} = -\sigma A T^4 \text{ or } \frac{dM}{dt} = -\frac{\alpha}{M^2}$$

where $\alpha = hc^6/(30720\pi^2 G^2)$. Integrating this separable differential equation

$$\int_{M_o}^{0} M^2 dM = -\alpha \int_{o}^{t} dt \implies t = \frac{M_o^3}{3\alpha}$$

which is the lifetime of a black hole.

(d) For a solar mass black hole the life time works out to be $t = 7 \times 10^{74}$ s which is almost 10^{60} times the age of the universe! Solar mass black holes are not going to evaporate anytime soon.

This long life time is now used to build models of cold dark matter. Maybe the dark matter component of the universe is composed from black holes that formed in the first moments after the big bang. They would still be around today.

(e) Working backward from the age of the universe of 13.7 billion years the mass of a primordial black hole that is evaporating now

$$M_o = (3\alpha t)^{(1/3)} \simeq 1 \times 10^{11} \text{ kg}$$

which is tiny as compared to a planet. Radiation from a black hole this size would be

$$\lambda_{peak} \simeq 15.9\frac{2GM}{c^2} \simeq 2.4 \times 10^{-15} \text{ m}.$$

This is really short and corresponds to a hard gamma ray. This is the peak wavelength of radiation that appears right after the black hole is formed. The radiation just gets more intense (and diverse) from there.

(6) Phase changes of bubbles:

(a) Ignoring surface tension to start, the Gibbs free energy should be a sum of the two phases, liquid l and vapor g, so

$$G = \mu_l N_l + \mu_g N_g = \frac{4\pi\mu_l r^3}{3v_l} + \mu_g N_g$$

where

$$N_l = \frac{4\pi r^3}{3v_l}$$

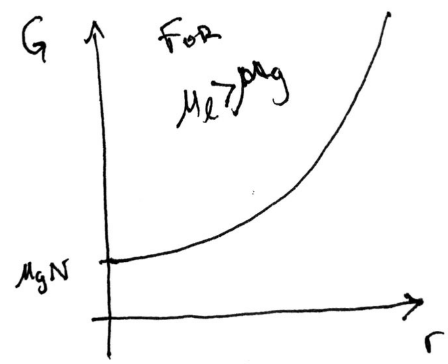
For later it is handy to express the Gibbs free energy in terms of the total number of particles $N = N_l + N_g$,

$$G = \mu_g N + \frac{4\pi r^3}{3v_l}(\mu_l - \mu_g).$$

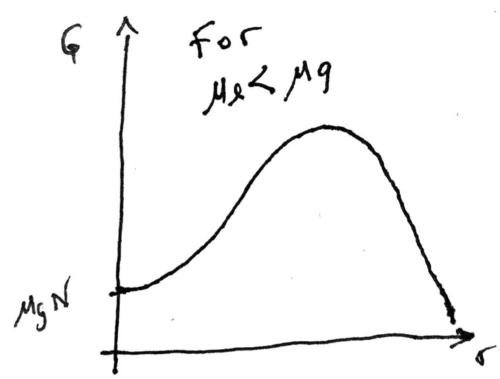
(b) The surface tension contributes to G via σA so the Gibbs free energy is now

$$G = \mu_g N + \frac{4\pi r^3}{3v_l}(\mu_l - \mu_g) + 4\pi r^2 \sigma.$$

(c) For $\mu_l > \mu_g$ we have both positive terms so G just grows with r,



For $\mu_l < \mu_g$ the r^3 term now dominates at large r and we get a bit of a bump



So we get an equilibrium in the $\mu_l < \mu_g$ case but it is not stable. That's ok maybe it runs away and condensation occurs! I'll call the equilibrium radius r_*.

(d) Finding r_*,

$$0 = \frac{dG}{dr} = \frac{4\pi r^2}{v_l}(\mu_l - \mu_g) + 8\pi r\sigma \implies r_* = \frac{2v_l\sigma}{\mu_g - \mu_l}$$

Ok, now for the difference in chemical potentials. Equation (5.40) gives us

$$\mu_g = \mu_o + kT\ln(P_g/P_o)$$

where the "o" 's are some reference state. Let's use the reference to be the vapor pressure of the surface liquid (when its flat). In this case the ratio of the pressures is the relative humidity (RH) and $\mu_o = \mu_l$. The relative humidity more of less maxes out at 100% or 1 but we'll consider higher RH's. Now

$$\mu_g - \mu_l = kT\ln(RH) \text{ and } r_* = \frac{2v_l\sigma}{kT\ln(RH)}.$$

Great. We are nearing something we can compute. Switching to a per mole basis the fraction

$$\frac{2v_l\sigma}{kT} = \frac{2V_l\sigma}{RT}$$

where V_l is the molar volume of water, about 18 mL. Therefore at 20°C ,

$$\frac{2V_l\sigma}{RT} = \frac{2\cdot 18\times 10^{-6}m^3\cdot 0.073J/m^2}{8.31J/K\cdot 293K} \simeq 1.08 \text{ nm}$$

This sets the scale. The radius is

$$r_* \simeq \frac{1.08 \text{ nm}}{\ln(RH)}$$

so this diverges for $RH = 1$ and then rapidly shrinks as the RH increases. So at least with assumptions of this model, small droplets do not form simply by spontaneous nucleation. It lokks like we need a center - e.g. dust to make this happen.