PHYS 370 Guide 3 Solutions

Fall 2024 v1.5

Solutions:

(1) Ready Team One?

(a) The probability that team two wins can be worked out case by case. Using the "most H's wins" version the probability is 1/2, which many of you found.
Alternatively, for the pairwise matching (as what follows in the later parts) we have, for instance, if team two flips "HH", which occurs with probability $P(HH) = 1/4$, then team two always wins. We can write as $P(2wins|HH) = 1$. We can then go through the cases and sum up the result,

$$\begin{aligned} P_2 &= P(HH)P(2wins|HH) + P(HT)P(2wins|HT) + P(TH)P(2wins|TH) \\ &+ P(TT)P(2wins|TT) \\ &= \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{8} = \frac{17}{32} \end{aligned}$$

(b) Since there are draws in the game let's compute this directly (rather than $P_1 = 1 - P_2$).

$$\begin{aligned} P_1 &= P(HHH)P(1wins|HHH) + P(HHT)P(1wins|HHT) \\ &+ P(TTH)P(1wins|TTH) + P(TTT)P(1wins|HH) \\ &= \frac{1}{8} \cdot \frac{1}{4} + \frac{3}{8} \cdot \frac{1}{4} + \frac{3}{8} \cdot 0 + \frac{1}{8} \cdot 0 = \frac{1}{8} \end{aligned}$$

So this is a crummy game for team one!

(c) Team one wins 2 units when either the result is "HHH" or the result is one of the two-head flips like "HHT", which occur with probability 1/2, *and* when team two flips "TT", which occurs with probability 1/4. Thus, team one wins 2 units with probability

$$P(1wins2) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

(d) The expectation value for team one is

$$\langle q \rangle = \sum_s q_s P_s = 2 \cdot P(1wins2) + 1 \cdot P(1wins1) + (-1)P(1wins - 1) + (-2)P(1wins - 2)$$

We computed $P(1wins2)$ in the last part so we have to compute the probability that team one wins 1 unit and looses 2 units. For the "wins 1 unit" case we can list the cases. However for every "wins 1" instance there is simultaneously a "looses 1" instance. They are all draws.
The probability of team one loosing 2 is found from all the two head flips for team one (P = 1/2) and the two head flips for team two (P=1/4) plus the probability of team one flipping "TTT" (P=1/8), when it always looses, plus the single head flips for team one and the "HH", "TH", and "HT" flips for team two. So,

$$P(1wins - 2) = \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{8} + \frac{3}{8} \cdot \frac{3}{4} = \frac{17}{32}.$$

Thus,

$$\langle q \rangle = 2 \cdot \frac{1}{8} - 2 \cdot \frac{17}{32} = -\frac{13}{16} = -0.8125.$$

(2) A series of questions on cooking pasta

(a) Here we assume that there is not significant heat flow (so $Q_{stove} = 0$) during the "pasta cools the water" period of the pasta cooking process. Therefore we have

$$Q_{\text{water}} + Q_{\text{pasta}} = 0$$

In each case the heat comes from a change in temperature, $Q = mc\Delta T$. Thus,

$$m_p c_p \left(T_f - T_i\right) = -m_{\text{water}} c_{\text{water}} \left(T_f - T_{\text{water}_i}\right)$$

where T_f is the final temperature of the pasta-water mixture; $T_i = 68°\text{F} = 20°\text{C} = 293$ K is the initial temperature of the pasta; and $m_{\text{water}} = 2000$ g is the mass of the water. Sorting out the algebra I find

$$T_f = \frac{m_{\text{water}} c_{\text{water}} \cdot 373 + m_p c_p \cdot 293}{m_{\text{water}} c_{\text{water}} + m_p c_p} \simeq 367 \text{ K} = 94.4°\text{C}$$

Apparently (although see the soaking method for another take!) pasta goes soggy when the temperature dips below about 80°C so this looks pretty good. The water temperature drops about 6°C .

(b) Using the Joy of Cooking method requires 7 quarts of tap water or about 6.6 liters of 287 K ($\simeq$ 57°F) water. Using "$Q = mc\Delta T$" for this water we obtain about 2.4×10^6 J.

(c) ...

You can explore cooking pasta more in the additional parts. Here's how the points work: Part (b) 0.5 pts. Part (c) 1 pt. For part (d) 0.5 pt. Part (e) 1 pt. I'll accept these solutions anytime before the last week in the semester.

(3) Hiking Marcy

(a) We each need $W = U = mgh$ to climb the mountain. My mass is approximately 78 kg. Running the conversions I find about 755 kJ required to do this work. Using 4184 J/kcal and the given efficiency I find I should eat 3.7 bowls of museli (a big breakfast!). This amounts to $Q \simeq 5.8 \times 10^6$ J or about 1400 kcal. Your answer will scale with your weight.

(b) Using a specific heat of 4.2 J/g °C and the relation $Q = mc\Delta T$, I find a temperature difference of $\Delta T = 13.3$ °C . Yikes! This is too much. (The temperature difference will not depend on your mass.)

(c) Using the given latent heat, I find

$$\frac{0.75 \cdot 5.8 \times 10^6 \text{ J}}{4184 \text{ J/kcal } \cdot 580 \text{ kcal/kg}} \simeq 1.8 \text{ liters of water.}$$

(This does scale with mass.) The metabolism of the cereal produces a bit of water but bringing enough water, or filtering it on the way, is safest. I've been on this route on hot summer days, cooler days and in the winter at about 10 °F . My water use differed drastically, with winter being the lowest.

(4) 20 coin flips

(a) For twenty two state systems

$$\Omega = 2^{20} = 1048576 \simeq 10^6$$

(b) and so the probability of any one state is

$$P = \frac{1}{2^{20}} \simeq 10^{-6}$$

(c) As we discussed in class this is like choosing to flip over 12 out of 20 coins, so for this "12" state

$$\Omega_{12} = \binom{20}{12} = \frac{20!}{12!\,8!} = 125970$$

so then the probability is $125970/2^{20} \simeq 12\%$.

(5) An $N = 30$ Einstein solid: For $N = 30 = q$

$$\Omega(30,30) = \binom{59}{30} = \frac{59!}{30!\,29!} \simeq 5.9 \times 10^{16}$$

Mathematica or Wolfram Alpha computes these easily.

(6) Two Einstein solids $N_A = 10 = N_B$ and $q = 20$:

(a) The macrostates are determined by q_A or q_B so since there are 21 values for one of these (0 - 20) there are 21 macrostates.

(b) We can find this with the combined system since the number of microstates of the whole system is equal to the number of microstates of the whole system The total number of microstates of the combined solid is

$$\Omega = \binom{20+20-1}{20} = \frac{39!}{20!\,19!}$$

Using Stirling's approximation

$$\Omega \simeq \frac{39^{39}e^{-39}}{20^{20}e^{-20}19^{19}e^{-19}} \simeq 6.9 \times 10^{10}$$

You could also find this from summing the the product of Ω_A and Ω_B over all macrostates.

(c) If all the energy is in solid A then

$$\Omega_* = \Omega_A\Omega_B = \binom{29}{20} \cdot 1 = \frac{29!}{20!\,9!} \simeq 1.0 \times 10^7$$

The probability is then

$$P = \frac{\Omega_*}{\Omega} \simeq 1.45 \times 10^{-4}$$

and so not so likely.

(d) For an even split

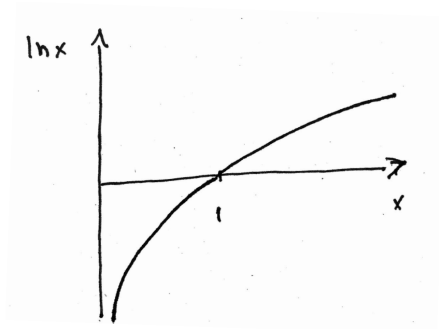
$$P = \frac{\Omega_A\Omega_B}{\Omega} = \frac{\binom{10+10-1}{10}^2}{6.9 \times 10^{10}} \simeq \frac{8.5 \times 10^9}{6.9 \times 10^{10}} \simeq 12.4\%$$

Much higher, yes, but still not super likely.

(e) If the energy units were unequally divided the behavior of the system as it evolves to equilibrium would be irreversible. In this case heat would flow between the systems.

(7) (Optional 1pt.) Log review

(a) Here's a sketch



(b) To prove these, use the given definition $e^{\ln x} = x$.

$$e^{\ln ab} = ab \text{ and } e^{\ln a + \ln b} = e^{\ln a} e^{\ln b} = a \cdot b \text{ so } \ln ab = \ln a + \ln b$$

Likewise,

$$e^{\ln a^b} = a^b \text{ and } e^{b \ln a} = \left(e^{\ln a}\right)^b = a^b \text{ so } \ln a^b = b \ln a$$

(c) Since

$$\frac{d}{dx} e^{\ln x} = e^{\ln x} \frac{d}{dx} \ln x = x \frac{d}{dx} \ln x$$

and

$$\frac{d}{dx} x = 1 \text{ then } x \frac{d}{dx} \ln x = 1 \implies \frac{d}{dx} \ln x = \frac{1}{x}$$

as desired.

(d) We did this in class on September 17. Use Taylor series to show

$$\ln(1+x) = \ln(1) + x + \ldots \text{ so } \ln(1+x) \simeq x.$$

For $x = 0.1$ the fractional error $\delta x / x$ in the approximation is about 5% while for $x = 0.01$ the approximation is 0.5%.