PHYS 370 Guide 4 Solutions

Fall 2024 v1.0

Solutions:

(1) Let me call the distance that the sugar syrup diffuses, Δy. The syrup is basically sucrose so $D = 5 \times 10^{-10}$ m^2/s. From the second form of the diffusion equation, equation (1.71), we have

$$\frac{N}{A\Delta t} = D\frac{N/V}{\Delta y}$$

With $V = A\Delta x$, canceling factors, and solving for the distance,

$$\Delta y = \frac{D\Delta t}{\Delta x} \simeq 3\ \mu\text{m, where } \Delta x = 1 \text{ cm.}$$

(2) A 10^3 coin flip:

(a) These are pretty big numbers so I used Stirling's approximation

$$\Omega(500) = \binom{1000}{500} \simeq 2^{10^3}\frac{1}{\sqrt{500\pi}}$$

The probability is this multiplicity divided by the total number of states,

$$\Omega_{tot} = 2^{10^3},$$

so

$$P(500) = \frac{\Omega(500)}{\Omega_{tot}} \simeq \frac{1}{\sqrt{500\pi}}.$$

This precise outcome is not very likely - just about 2%.

(b) For 600 heads

$$\Omega(600) = \binom{1000}{600} \simeq \frac{1000^{1000}}{600^{600}400^{400}\sqrt{480\pi}}$$

The probability is this multiplicity divided by the total number of states, so

$$P(600) = \frac{\Omega(600)}{\Omega_{tot}} \simeq \frac{500^{1000}}{600^{600}400^{400}\sqrt{480\pi}} \simeq 4.6 \times 10^{-11}.$$

Mathematica was handy to use for the final computation. This probability is *tiny* compared to $P(500)$.

(3) For an Einstein solid in the low temperature limit, $N \gg q \gg 1$, the multiplicity is

$$\Omega = \binom{q+N-1}{q} \simeq \frac{(q+N)!}{q!N!}.$$

So the log is, using Stirling's approximation,

$$\ln\Omega \simeq (q+N)\ln(q+N) - q\ln q - N\ln N$$

which is also equ'n (2.18). Now expanding the log

$$\ln(q+N) = \ln q\left(1 + \frac{q}{N}\right) \simeq \ln q + \frac{q}{N}.$$

Expanding, dropping the relatively small q^2/N term gives

$$\ln\Omega \simeq q\left(\ln\frac{q}{N}\right) + q.$$

So that, once we exponentiate

$$\Omega \simeq \left(\frac{Ne}{q}\right)^q$$

which is eq'n (2.21) with q and N exchanged.

(4) Two Einstein solids. Use the table in Figure 2.5 to obtain

$$S_{max} = k \ln \Omega_A(60)\Omega_B(40) \simeq \ln(6.9 \times 10^{114}) \simeq 264k$$

and

$$S_{min} = k \ln \Omega_A(0)\Omega_B(100) \simeq \ln(2.8 \times 10^{81}) \simeq 188k$$

The sum over all possibilities gives

$$S_{all} = k \ln(9.3 \times 10^{115}) \simeq 267k$$

only a bit larger than for the 'max' value above. As for 'long time scales' we expect that the system will end up in the maximum entropy state. For even longer time scales the system might explore more states.

(5) We did much of this in class. Starting with (2.40)

$$\Omega_N \sim \frac{1}{N!}\frac{V^N}{h^{3N}}\frac{1}{(3N/2)!}\left(\sqrt{2\pi mU}\right)^{3N}$$

we use Stirling's approximation on the factorials and gather powers of N to obtain

$$\Omega_N \simeq \left(\frac{Ve}{N}\frac{e^{3/2}}{(3N/2)^{3/2}}\frac{\left(\sqrt{2\pi mU}\right)^3}{h^3}\right)^N .$$

Now taking the natural log, gathering factors to the 3/2 power, and collecting powers of e we find

$$\frac{S}{k} = \ln \Omega_N = N\left\{\ln\left[\left(\frac{V}{N}\right)\left(\frac{4\pi mU}{3Nh^2}\right)^{3/2}\right] + \frac{5}{2}\right\}$$

which is the Sakur-Tetrode equation.

(6) Estimates of entropy: Using the general plan of $S \sim Nk$, a book is 1 kg and the molar mass of C is 12 g so

$$N_{book} \sim \frac{1kg}{.012kg} \cdot 10^{23} \sim \text{ and so } S \sim 10^{26}k \sim 10^3 \text{ J/K}$$

Likewise,

$$N_{moose} \sim \frac{400kg}{.018kg} \cdot 10^{23} \sim \text{ and so } S \sim 10^{29}k \sim 10^6 \text{ J/K}$$

and

$$N_{sun} \sim \frac{10^{30}kg}{.001kg} \cdot 10^{23} \sim \text{ and so } S \sim 10^{56}k \sim 10^{33} \text{ J/K}$$

if there is no additional contribution from the particles themselves. But there probably is, given the high temperature. So we might expect a somewhat higher entropy.

(7) Processes and more processes... In each case the number of configurations increases; the multiplicity increases, and the entropy increases.

(a) Salt dissolves. The dissociated Na and Cl can roam more widely so Ω goes up and entropy increases.

(b) Proteins denature. The long chains can now flop around in more ways so Ω goes up and entropy increases.
(c) The structure breaks up. There are many ways to break so Ω goes up and entropy increases.
(d) As in (c)
(e) The are multiple angles to fall over.
(f) During combustion the molecule dissociates, adding more states, and there is also of heat released. This can go to lots of degrees of freedom.

(8) Black holes!
(a) From Newton's law of gravitation
$$F = -\frac{GmM}{r^2}$$
we can find that the dimensions of G are
$$[G] = \frac{L^3}{MT^2}$$
where I am using M for mass, T for time, and L for length. To find an expression of length we need a mass and a factor of c^2;
$$R \sim \frac{GM}{c^2}$$
works. For a one solar mass black hole, this R is about 1.5 km. The actual radius of the horizon is the Schwarzschild radius,
$$R_S = \frac{2GM}{c^2}.$$
A solar mass black hole has a radius then of about 3 km.
(b) If the black hole is formed by collapsing N particles then the particles' entropy is $\sim Nk$ and, once they collapse and form a black hole, it presumably has entropy $S_{BH} \sim Nk$. Here N is the maximum number of particles that could be used to form the BH at a fixed mass.
(c) For that maximum - I'll come back to this in a moment - let's choose N low energy photons each with a wavelength of about the horizon radius, $\lambda \sim GM/c^2$. For one of these photons,
$$E_\lambda = \frac{hc}{\lambda} \sim \frac{hc^3}{GM}$$
and the total would be
$$E_{tot} = Mc^2 = NE_\lambda \implies Mc^2 \sim N\frac{hc^3}{GM}$$
so the number of particles is
$$N \sim \frac{GM^2}{hc}$$
Hence,
$$S \sim Nk \sim \frac{kG}{hc}M^2$$
which, apart from a factor of $8\pi^2$, is the Bekenstein-Hawking entropy of a black hole! This is proportional to the area of the horizon of the black hole. Now returning to the "maximum" mentioned above, suppose the photons were higher energy, say had half the wavelength of the radius. Then E_λ would double and N would be half what is was. Thus

the entropy would also be half the size. So the horizon-sized wavelengths seems about right for the maximum. (There remains an interesting question of how these incoming photons would appear far away. But this is a GR question.)

(d) Running the numbers for a solar mass BH gives $S_{BH} \simeq 1.5 \times 10^{54}$ J/K. That is big! The above estimate for the sun indicates an an entropy of $\sim 10^{33}$ J/K.

(e) If all these degrees of freedom where on the horizon in a number of geometric particles, N_{geom}, then, since $A = 4\pi R_S^2 = 16\pi G^2 M^2/c^4$, the BH entropy is

$$S_{BH} \sim \frac{kG}{hc} M^2 \sim k \frac{A}{\ell^2}$$

for $N_{geom} = A/\ell^2$ some length scale ℓ. Substituting the BH entropy we can calculate this length

$$\ell = \sqrt{\frac{Gh}{c^3}}$$

a version of the Planck length, about 10^{-35} m - tiny! So $\ell^2 \sim 10^{70}$ m^2.