Stat Mech (PHYS 370) Guide 8 Solutions

Fall 2024 v1.0

Problems:

(1) (a) For a harmonic oscillator the partition function is

$$Z = \sum_s e^{-\beta E_s} = \sum_{n=0}^{\infty} e^{-\beta h\nu(n+\frac{1}{2})}$$

We'll compute Z at $T = 300$ K first. At that temperature, and with the given frequency, $\beta h\nu \simeq 7.68$, and

$$Z \simeq e^{-7.68/2} + e^{-(3/2)\,7.68} + e^{-(5/2)\,7.68} \simeq 0.0215$$

where the first term is dominant. Therefore it is not surprising that

$$P_1 = \frac{e^{-\beta h\nu(3/2)}}{Z} \simeq 4.6 \times 10^{-4} \text{ and } P_2 \simeq 2.1 \times 10^{-7}$$

while for the ground state

$$P_0 \simeq \frac{e^{-7.68/2}}{Z} \simeq 0.99954 \simeq 1$$

Just about all the H_2O's are not vibrating in this hinge mode at this temperature.

(b) At at the higher temperature $T = 700$ K all we have to do is recompute the same quantities. The combination $\beta h\nu \simeq 3.25$ and $Z \simeq 0.205$. Likewise,

$$P_0 \simeq \frac{e^{-3.25/2}}{Z} \simeq 0.96,\ P_1 \simeq 0.038, \text{ and } P_2 \simeq 0.001$$

so a bit under 4% of the water molecules will be excited in this mode.

(2) Some care is needed in reading his description. The description of the populations gives us a relative probability. Calling the first excited state 1 and the ground state 0, we have - recalling the degeneracy of 3 -

$$\frac{P(1)}{P(0)} = \frac{3}{10} = \frac{3e^{-\beta E(1)}}{e^{-\beta E(0)}} \implies \frac{1}{10} = e^{-\beta \Delta E}$$

where the difference in energy levels is $\Delta E = E(1) - E(0)$. Hence,

$$T = \frac{\Delta E}{k(\ln 10)} \simeq 2.4 \text{ K}$$

which isn't far off the temperature of the microwave background at 2.7 K.

(3) The relative abundance is determined by the Boltzmann weights, and the temperature, of course. Calling neutrons "n" and protons "p",

$$\frac{P(n)}{P(p)} = \frac{e^{-\beta E(n)}}{e^{-\beta E(p)}} = e^{-\beta \Delta E}.$$

We are given the difference in energy levels is $\Delta E = E(n) - E(p) = (2.3 \times 10^{-30})c^2$. Hence,

$$\frac{P(n)}{P(p)} \simeq 86$$

so there are 43 neutrons for every 50 protons. The mixture is 43/93 = 0.46 neutron and 50/93 = 0.54 proton.

(4) I think we did this computation in class but if so it was awhile ago... The average energy or expectation value is

$$\bar{E} \equiv \langle E \rangle = \sum_s E(s)P(s) = \sum_s E(s)\frac{e^{-\beta E(s)}}{Z}.$$

We want to obtain this via a derivative, which we can since

$$-\frac{\partial}{\partial \beta} e^{-\beta E(s)} = E(s)e^{-\beta E(s)}.$$

Thus,

$$\langle E \rangle = -\frac{1}{Z}\sum_s \frac{\partial}{\partial \beta} e^{-\beta E(s)} = -\frac{1}{Z}\frac{\partial}{\partial \beta}\sum_s e^{-\beta E(s)} = -\frac{1}{Z}\frac{\partial}{\partial \beta} Z = -\frac{\partial}{\partial \beta}\ln Z$$

as expected.

(5) Standard deviation and energy fluctuations

(a) The average is 3 eV so the deviations are -3 (two of these) 1 (two of these) and 4 eV.

(b) The average of the squares of the deviations is

$$\frac{36}{5} = 7.2$$

The standard deviation is the square root of this so about 2.7, which agrees with the deviations above.

(c) In general

$$\begin{aligned}\sigma^2 &= \frac{1}{N}\sum_i (\Delta E_i)^2 = \frac{1}{N}\sum_i (\bar{E} - E_i)^2 \\ &= \frac{1}{N}\sum_i (\bar{E}^2 - 2\bar{E}E_i + E_i^2)^2 \\ &= \frac{1}{N}\sum_i \bar{E}^2 - \frac{2}{N}\sum_i \bar{E}E_i + \frac{1}{N}\sum_i \bar{E}_i^2 \\ &= \langle E \rangle^2 - 2\langle E \rangle \langle E \rangle + \left\langle E^2 \right\rangle = \left\langle E^2 \right\rangle - \langle E \rangle^2\end{aligned}$$

where I switched from the over-bar to the expectation value notation for averages for clarity. This problem appears in statistics, quantum mechanics, and stat mech. Once you have done this in one context, you have done them all.

(d) From the numbers above $\left\langle E^2 \right\rangle = 16.2$ eV^2 while the square of the average energy of 3 eV is 9 eV^2 so

$$\sigma^2 = \left\langle E^2 \right\rangle - \langle E \rangle^2 = 16.2 - 9 = 7.2 \text{ eV}^2$$

as above.

(6) A black hole model:

(a) The key part is to expand the square root for large-ish j,

$$\sqrt{j(j+1)} = j\sqrt{1+\frac{1}{j}} \simeq j\left(1+\frac{1}{2j}\right) = \frac{1}{2}(2j+1)$$

I used $(1+x)^n \simeq 1 + nx$ in the middle step. This gives $E(j) = \epsilon m$ in the notation of the problem.

(b) By definition

$$Z_1 = \sum_j d_j e^{-\beta E(j)} = \sum_{j=1/2}^{\infty} (2j+1) e^{-\beta\epsilon(2j+1)} = \sum_{m=2}^{\infty} m e^{-\beta\epsilon m}.$$

as expected where $m = 2j + 1$.

(c) Letting $x = e^{-\beta\epsilon}$ we see that

$$x\frac{d}{dx}\sum_{m=2}^{\infty} x^m = \sum_{m=2}^{\infty} m x^m$$

while

$$\sum_{m=2}^{\infty} x^m = x^2 + x^3 + \cdots = x^2\left(1 + x + x^2 + \dots\right) = \frac{x^2}{1-x}.$$

so we can differentiate to find

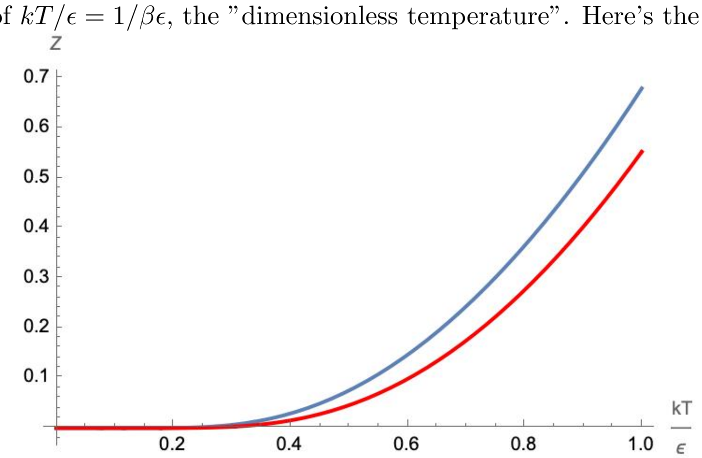
$$Z_1 = x\frac{d}{dx}\left(\frac{x^2}{1-x}\right) = \frac{2x}{1-x} - \frac{x^2(-1)}{(1-x)^2} = \frac{2-e^{-\beta\epsilon}}{\left(e^{\beta\epsilon}-1\right)^2}$$

as expected.

(d) To compare the two partition functions I plotted Z_1 above and

$$Z_{exact} = \sum_{j=1/2}^{\infty} (2j+1) e^{-\beta\epsilon\sqrt{j(j+1)}}$$

as functions of $kT/\epsilon = 1/\beta\epsilon$, the "dimensionless temperature". Here's the plot:



The two partitions diverge when

$$\frac{kT}{\epsilon} \simeq 0.2 \implies T = \frac{0.1 g\hbar}{kc},$$

which is dependent on the acceleration. Near the horizon the acceleration is large so the temperature is also large.

(e) With N indistinguishable particles of geometry the partition function is (approximately)

$$Z_N = \frac{Z_1^N}{N!}$$

as we saw in class. Thus,

$$Z_N = \frac{1}{N!}\left(\frac{2-e^{-\beta\epsilon}}{\left(e^{\beta\epsilon}-1\right)^2}\right)^N \simeq \left(\frac{\left(2-e^{-\beta\epsilon}\right)e}{N\left(e^{\beta\epsilon}-1\right)^2}\right)^N$$

(7) The partition function for this system is

$$Z = \frac{Z_1^N}{N!}.$$

The Helmholtz free energy is then

$$F = -kT\ln Z = -kT\left(N\ln Z_1 - \ln N!\right)$$

or, with Stirling's approximation,

$$F = -kT\left(N\ln Z_1 - N\ln N + N\right) = -NkT\left(\ln\frac{Z_1}{N} + 1\right).$$

The chemical potential is then

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} = -kT\left(\ln\frac{Z_1}{N} + 1\right) + NkT\frac{N}{Z_1}\frac{Z_1}{N^2} = -kT\ln\frac{Z_1}{N}.$$

(8) For the relativistic case: When the momentum is high we have

$$E = \sqrt{(pc)^2 + m^2c^4} = (pc)\sqrt{1 + \frac{m^2c^2}{p^2}} \simeq pc + \frac{1}{2}\frac{m^2c^3}{p}$$

where I have used the beginning of the binominal theorem, $(1+x)^n \simeq 1 + nx$, which works well when x is small. In this case if $mc/p < 1$ then the approximation works well.

Using the relativistic energy $E(n) = p(n)c$ and the de Broglie wavelength $p_n = hn/2L$ to find $E(n) = hnc/2L$ the single particle partition function in 1D is

$$Z_1 = \sum_n e^{-\beta E(n)} = \sum_n e^{-\beta hnc/2L} \simeq \int_0^\infty dn\, e^{-\beta hnc/2L} = \frac{2LkT}{hc}$$