

- (1) Harmonic oscillators
 (a) He's a start on the long division

$$\begin{array}{r}
 1 + x + x^2 + \dots \\
 1-x \overline{) 1} \\
 \underline{1-x} \\
 x \\
 \underline{x-x^2} \\
 x^2 \\
 \underline{x^2-x^3} \\
 x^3
 \end{array}$$

where the second lines are subtracted from the first lines, as one would do in ordinary long division.

- (b) Using $\epsilon = h\nu = \hbar\omega$, the single oscillator partition function is the sum over states

$$Z_1 = \sum_s e^{-\beta E(s)} = 1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon} + \dots$$

but this is just the sum in part (a) so

$$Z_1 = \frac{1}{1 - e^{-\beta\epsilon}}.$$

Note you may have (reasonably enough!) started with the energy levels

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right)$$

in which case you would have obtained

$$Z_1 = [2 \sinh(\beta\epsilon/2)]^{-1}$$

after doing similar algebraic steps.

- (c) Differentiating to obtain interesting quantities

$$\langle E \rangle_1 = -\frac{\partial \ln Z_1}{\partial \beta} = \frac{\epsilon e^{-\beta\epsilon}}{1 - e^{-\beta\epsilon}} = \frac{\epsilon}{e^{\beta\epsilon} - 1}.$$

- (d) The total energy of N identical, independent oscillators is just the sum of these energies for each one,

$$U = N \langle E \rangle = \frac{N\epsilon}{e^{\beta\epsilon} - 1}$$

which was what was in the solution to 3.25. You should look back at this to see how much easier this method is.

If you had started with the hyperbolic sine form of the partition function then the average energy is

$$U = N \langle E \rangle = N \coth(\beta\epsilon/2)$$

- (e) The rest of this is as before in the solution of problem 7 in Guide 6.

2

- (2) This is similar to the intro to mathematica - now with a realistic case of an oscillator with one anharmonic term. Here's the mathematica file I used:

(* Solution to 6.21 - anharmonic oscillator model for H2. k=1 *)

$$\text{In[*]}:= Z = \text{Sum}[\text{Exp}[-\beta * \epsilon * (1.03 * n - .03 * n^2)], \{n, 0, 11\}]$$

$$\text{Out[*]}= 1. + e^{-7.7 \beta \epsilon} + e^{-7.3 \beta \epsilon} + e^{-6.84 \beta \epsilon} + e^{-6.32 \beta \epsilon} + e^{-5.74 \beta \epsilon} + e^{-5.1 \beta \epsilon} + e^{-4.4 \beta \epsilon} + e^{-3.64 \beta \epsilon} + e^{-2.82 \beta \epsilon} + e^{-1.94 \beta \epsilon} + e^{-1. \beta \epsilon}$$

(* Single molecule partition function - and so N=1 *)

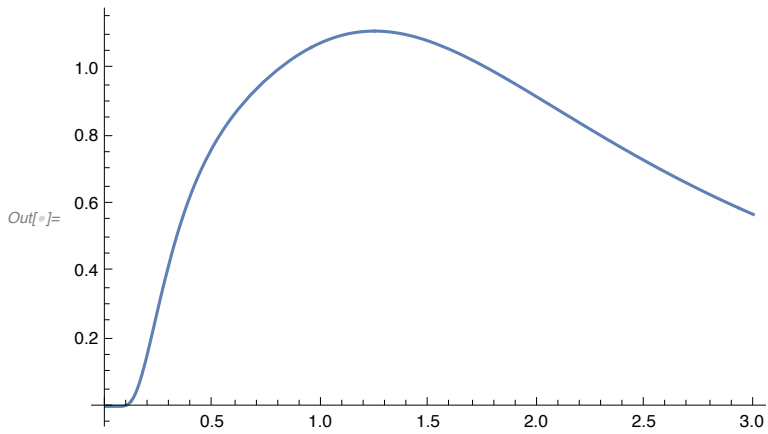
$$\text{In[*]}:= U = - (1 / Z) * D[Z, \beta] /. \beta \rightarrow 1 / T$$

$$\text{Out[*]}= - \left(\left(-7.7 e^{-\frac{7.7 \epsilon}{T}} - 7.3 e^{-\frac{7.3 \epsilon}{T}} - 6.84 e^{-\frac{6.84 \epsilon}{T}} - 6.32 e^{-\frac{6.32 \epsilon}{T}} - 5.74 e^{-\frac{5.74 \epsilon}{T}} - 5.1 e^{-\frac{5.1 \epsilon}{T}} - 4.4 e^{-\frac{4.4 \epsilon}{T}} - 3.64 e^{-\frac{3.64 \epsilon}{T}} - 2.82 e^{-\frac{2.82 \epsilon}{T}} - 1.94 e^{-\frac{1.94 \epsilon}{T}} - 1. e^{-\frac{1. \epsilon}{T}} \right) / \left(1. + e^{-\frac{7.7 \epsilon}{T}} + e^{-\frac{7.3 \epsilon}{T}} + e^{-\frac{6.84 \epsilon}{T}} + e^{-\frac{6.32 \epsilon}{T}} + e^{-\frac{5.74 \epsilon}{T}} + e^{-\frac{5.1 \epsilon}{T}} + e^{-\frac{4.4 \epsilon}{T}} + e^{-\frac{3.64 \epsilon}{T}} + e^{-\frac{2.82 \epsilon}{T}} + e^{-\frac{1.94 \epsilon}{T}} + e^{-\frac{1. \epsilon}{T}} \right) \right)$$

$$\text{In[*]}:= Cv = D[U, T] /. T \rightarrow t * \epsilon$$

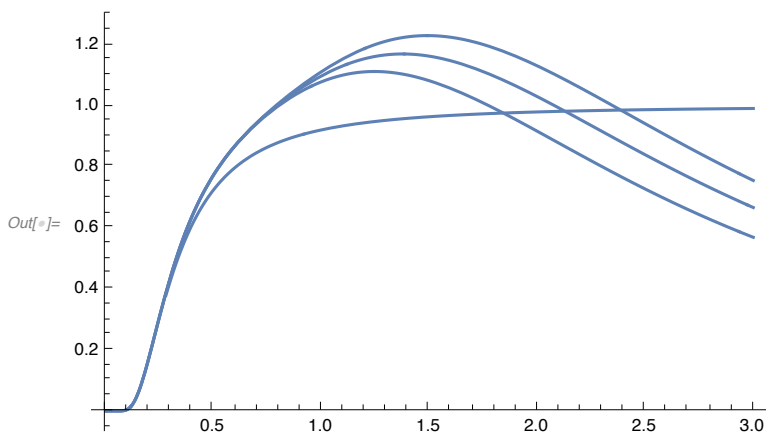
$$\text{Out[*]}= - \left(\left(- \frac{59.29 e^{-7.7/t}}{t^2} - \frac{53.29 e^{-7.3/t}}{t^2} - \frac{46.7856 e^{-6.84/t}}{t^2} - \frac{39.9424 e^{-6.32/t}}{t^2} - \frac{32.9476 e^{-5.74/t}}{t^2} - \frac{26.01 e^{-5.1/t}}{t^2} - \frac{19.36 e^{-4.4/t}}{t^2} - \frac{13.2496 e^{-3.64/t}}{t^2} - \frac{7.9524 e^{-2.82/t}}{t^2} - \frac{3.7636 e^{-1.94/t}}{t^2} - \frac{1. e^{-1./t}}{t^2} \right) / \left(1. + e^{-7.7/t} + e^{-7.3/t} + e^{-6.84/t} + e^{-6.32/t} + e^{-5.74/t} + e^{-5.1/t} + e^{-4.4/t} + e^{-3.64/t} + e^{-2.82/t} + e^{-1.94/t} + e^{-1./t} \right) \right) + \left(\left(\frac{7.7 e^{-7.7/t}}{t^2 \epsilon} + \frac{7.3 e^{-7.3/t}}{t^2 \epsilon} + \frac{6.84 e^{-6.84/t}}{t^2 \epsilon} + \frac{6.32 e^{-6.32/t}}{t^2 \epsilon} + \frac{5.74 e^{-5.74/t}}{t^2 \epsilon} + \frac{5.1 e^{-5.1/t}}{t^2 \epsilon} + \frac{4.4 e^{-4.4/t}}{t^2 \epsilon} + \frac{3.64 e^{-3.64/t}}{t^2 \epsilon} + \frac{2.82 e^{-2.82/t}}{t^2 \epsilon} + \frac{1.94 e^{-1.94/t}}{t^2 \epsilon} + \frac{1. e^{-1./t}}{t^2 \epsilon} \right) \left(-7.7 e^{-7.7/t} \epsilon - 7.3 e^{-7.3/t} \epsilon - 6.84 e^{-6.84/t} \epsilon - 6.32 e^{-6.32/t} \epsilon - 5.74 e^{-5.74/t} \epsilon - 5.1 e^{-5.1/t} \epsilon - 4.4 e^{-4.4/t} \epsilon - 3.64 e^{-3.64/t} \epsilon - 2.82 e^{-2.82/t} \epsilon - 1.94 e^{-1.94/t} \epsilon - 1. e^{-1./t} \epsilon \right) \right) / \left(1. + e^{-7.7/t} + e^{-7.3/t} + e^{-6.84/t} + e^{-6.32/t} + e^{-5.74/t} + e^{-5.1/t} + e^{-4.4/t} + e^{-3.64/t} + e^{-2.82/t} + e^{-1.94/t} + e^{-1./t} \right)^2$$

```
In[*]:= p3 = Plot[Cv, {t, 0, 3}]
```



(* I recomputed this plot for $n_{\max} = 15, 13,$ and 11 in descending order and shown in the following plot with the usual harmonic case *)

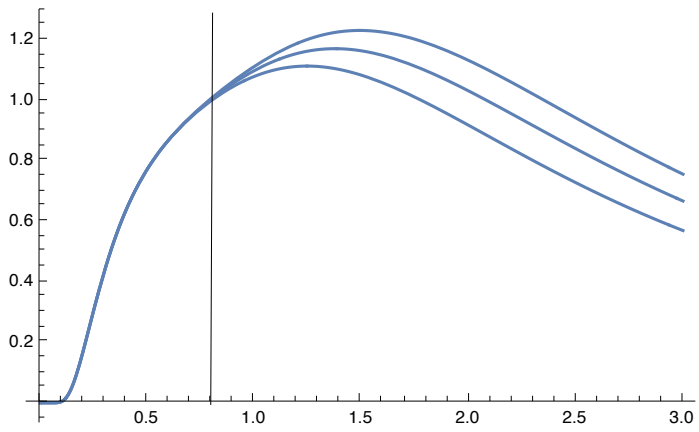
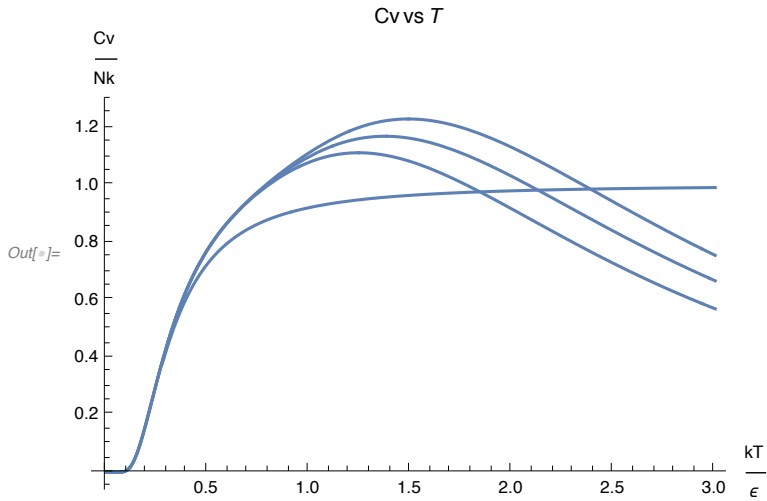
```
In[*]:= Show[p1, p2, p3, p4]
```



```

In[ ]:= Show[%220, AxesLabel -> {HoldForm[ $\frac{kT}{\epsilon}$ ], HoldForm[ $\frac{Cv}{Nk}$ ]},
            PlotLabel -> HoldForm[Cv vs T], LabelStyle -> {GrayLevel[0]}]

```



The set of curves are in order from top down, $n_{max} = 15, n_{max} = 13, n_{max} = 11$.

You can see in the final plots that the partition functions with fewer terms differ dramatically at high temperature - but that is where the approximation fails. The key take away is that the anharmonic model's heat capacity rises above the harmonic case - just as shown in the "vibrational" portion of figure 1.13. We are told that the hydrogen dissociates before the heat capacity would level off so that anharmonic case shows the extra increase before dissociation, as we see in figure 1.13.

- (3) With two identical atoms in the oxygen molecule we can use equation 6.33

$$Z_{rot} \simeq \frac{kT}{2\epsilon} = 72$$

This is a good approximation at room temperature $kT \sim 1/40 \gg \epsilon = 1.8 \times 10^{-4}$ eV.

- (4) The rotational partition function has the first two terms of

$$Z = 1 + 3e^{-2\beta\epsilon} + \dots$$

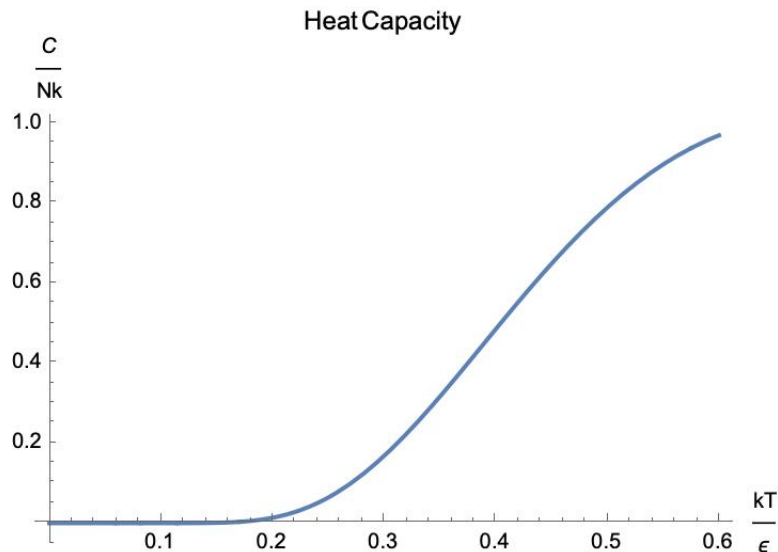
In the low temperature, $kT \ll \epsilon$, limit these are the largest terms. We'll keep only the leading order term. The average energy is

$$\langle E \rangle = \sum_j E(j)(2j+1)e^{-\beta E(j)} \simeq \frac{(2\epsilon)(3)e^{-2\beta\epsilon}}{1+3e^{-2\beta\epsilon}} \simeq 6\epsilon e^{-2\beta\epsilon}$$

where the last approximation recognizes that the denominator contributes only higher order terms. The heat capacity is then

$$C = \frac{\partial \langle E \rangle}{\partial T} = \frac{\partial \beta}{\partial T} \frac{\partial \langle E \rangle}{\partial \beta} \simeq \left(-\frac{1}{kT^2} \right) (-12\epsilon^2 e^{-2\beta\epsilon}) = 3k \left(\frac{2\epsilon}{kT} \right)^2 e^{-2\epsilon/kT}.$$

Other than the degeneracy, this is the same result as for a two state system - which makes sense given that we have truncated the system to two states. The plot of the heat capacity (C_v/Nk) vs. dimensionless temperature (kT/ϵ) is



It is a curve rising from $C_V = 0$ at $T = 0$ to a level line at high temp. The behavior of C_V around zero is consistent with the third law, $C_V \rightarrow 0$ as $T \rightarrow 0$. Including higher order terms would correct the result and interpolate between these two limiting cases.

- (5) The rotation partition function is

$$Z = \sum_j (2j + 1) e^{-\beta \epsilon_j(j+1)}$$

Here's the mathematica file,

(* Solution to 6.28 - quantum rotor model - here j is an integer! *)

In[16]:= Z = Sum[(2 * j + 1) * Exp[-β * ε * j (j + 1)], {j, 0, 6}]

Out[16]= $1 + 13 e^{-42 \beta \epsilon} + 11 e^{-30 \beta \epsilon} + 9 e^{-20 \beta \epsilon} + 7 e^{-12 \beta \epsilon} + 5 e^{-6 \beta \epsilon} + 3 e^{-2 \beta \epsilon}$

In[17]:= U = -(1 / Z) * D[Z, β]

Out[17]=
$$-\frac{546 e^{-42 \beta \epsilon} \epsilon - 330 e^{-30 \beta \epsilon} \epsilon - 180 e^{-20 \beta \epsilon} \epsilon - 84 e^{-12 \beta \epsilon} \epsilon - 30 e^{-6 \beta \epsilon} \epsilon - 6 e^{-2 \beta \epsilon} \epsilon}{1 + 13 e^{-42 \beta \epsilon} + 11 e^{-30 \beta \epsilon} + 9 e^{-20 \beta \epsilon} + 7 e^{-12 \beta \epsilon} + 5 e^{-6 \beta \epsilon} + 3 e^{-2 \beta \epsilon}}$$

In[18]:= U /. β → 1 / T

Out[18]=
$$-\frac{546 e^{-\frac{42 \epsilon}{T}} \epsilon - 330 e^{-\frac{30 \epsilon}{T}} \epsilon - 180 e^{-\frac{20 \epsilon}{T}} \epsilon - 84 e^{-\frac{12 \epsilon}{T}} \epsilon - 30 e^{-\frac{6 \epsilon}{T}} \epsilon - 6 e^{-\frac{2 \epsilon}{T}} \epsilon}{1 + 13 e^{-\frac{42 \epsilon}{T}} + 11 e^{-\frac{30 \epsilon}{T}} + 9 e^{-\frac{20 \epsilon}{T}} + 7 e^{-\frac{12 \epsilon}{T}} + 5 e^{-\frac{6 \epsilon}{T}} + 3 e^{-\frac{2 \epsilon}{T}}}$$

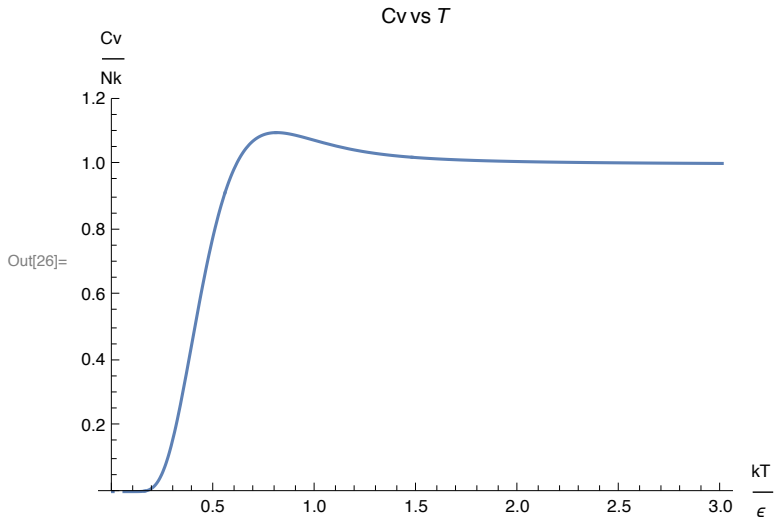
In[19]:= Cv = D[(U /. β → 1 / T), T] /. T → t * ε

Out[19]=
$$-\frac{\frac{22932 e^{-42/t} \epsilon}{t^2} - \frac{9900 e^{-30/t} \epsilon}{t^2} - \frac{3600 e^{-20/t} \epsilon}{t^2} - \frac{1008 e^{-12/t} \epsilon}{t^2} - \frac{180 e^{-6/t} \epsilon}{t^2} - \frac{12 e^{-2/t} \epsilon}{t^2}}{1 + 13 e^{-42/t} + 11 e^{-30/t} + 9 e^{-20/t} + 7 e^{-12/t} + 5 e^{-6/t} + 3 e^{-2/t}} +$$

$$\left(\left(\frac{546 e^{-42/t} \epsilon}{t^2 \epsilon} + \frac{330 e^{-30/t} \epsilon}{t^2 \epsilon} + \frac{180 e^{-20/t} \epsilon}{t^2 \epsilon} + \frac{84 e^{-12/t} \epsilon}{t^2 \epsilon} + \frac{30 e^{-6/t} \epsilon}{t^2 \epsilon} + \frac{6 e^{-2/t} \epsilon}{t^2 \epsilon} \right) \right. \\ \left. (-546 e^{-42/t} \epsilon - 330 e^{-30/t} \epsilon - 180 e^{-20/t} \epsilon - 84 e^{-12/t} \epsilon - 30 e^{-6/t} \epsilon - 6 e^{-2/t} \epsilon) \right) / \\ (1 + 13 e^{-42/t} + 11 e^{-30/t} + 9 e^{-20/t} + 7 e^{-12/t} + 5 e^{-6/t} + 3 e^{-2/t})^2$$

In[26]:= p4 = Plot[Cv, {t, 0, 3}, PlotRange → {0, 1.2},

AxisLabel → {HoldForm[$\frac{kT}{\epsilon}$], HoldForm[$\frac{Cv}{Nk}$]}, PlotLabel → HoldForm[Cv vs T]



(* compares values of the 7 terms in Z at kT/epsilon = 3 to see iif more terms would improve the calculation *)


```
In[24]:= N[Table[(2 * j + 1) Exp[- j (j + 1) / 3], {j, 0, 7}]]
```

```
Out[24]= {1., 1.54025, 0.676676, 0.128209, 0.0114537, 0.000499399, 0.0000108099, 1.1729 × 10-7}
```

(* Nope! The last term is already
small so more terms will not improve the plot *)

The takeaway in terms of the physics is the overshoot in C_V before leveling off at the “equipartition value” at high temperatures. Had we not done this problem and instead started with 6.26 I would not have anticipated this increase in C_V .

- (6) On Earth’s atmosphere: We are interested in the probability of molecules reaching escape velocity, about 11 km/s.
- (a) The probability is given in equation (6.54) and in a much more useful form in (6.55). The lower limit of the integral is given by

$$x_{min} = \frac{v_{esc}}{v_{max}},$$

where v_{esc} is the escape speed for the planet and v_{max} is the maximum of the Maxwell velocity distribution. From class,

$$v_{esc} = \sqrt{\frac{2GM_{\odot}}{r}} \simeq 1.1 \times 10^4 \text{ m/s}$$

for Earth. The maximum is

$$v_{max} = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2RT}{M}}.$$

For N_2 this is about 770 m/s. Computing the ratio gives $x_{min} \simeq 14.27$. Now using Mathematica and the command `NIntegrate[x^2*Exp[-x^2], {x, 14.27, Infinity}]` I find (multiplying by $4/\sqrt{\pi}$)

$$P_{N_2} = \frac{4}{\sqrt{\pi}} \text{NIntegrate}[x^2 \text{Exp}[-x^2], \{x, 14.27, \text{Infinity}\}] \simeq 5.9 \times 10^{-88}.$$

Not big ... but we don’t have much to compare to at this stage..

- (b) For hydrogen we do the same analysis but now $M = 2 \text{ g}$ so $v_{max} \simeq 2880 \text{ m/s}$ and $x_{min} \simeq 3.82$ giving $P_{H_2} \simeq 2.1 \times 10^{-6}$. Well, that is *a lot* larger. Not much H_2 in our atmosphere I would guess! For He, the computation is similar beginning with $M = 4 \text{ g}$. This yields $P_{He} \simeq 1.4 \times 10^{-12}$. Ok, smaller but still much larger than the probability for nitrogen. Much of the original hydrogen and helium escaped Earth’s atmosphere by now.

For CO_2 the molar mass is $M = 44 \text{ g}$ from our BOE climate change calculation. Thus, $v_{max} \simeq 615 \text{ m/s}$ and

$$x_{min} = \frac{v_{esc}}{v_{max}} \simeq \frac{11 \text{ km/s}}{615 \text{ m/s}} \simeq 17.9$$

giving $P_{CO_2} \simeq 8.2 \times 10^{-139}$. Well, that is *hugely* smaller. So not much CO_2 is going to leak out of our atmosphere. The excess we produce will be around for a while.

- (c) Now moving to the (newly formed) moon. Suppose it had a similar atmosphere with nitrogen at the same temperature. Now the escape velocity from the moon is about

$$v_{esc} = \sqrt{\frac{2GM}{R}} \simeq 2.4 \text{ km/s}.$$

So, $x_{min} = v_{esc}/v_{max} \simeq 2400/770$ and $P \simeq 2.2 \times 10^{-4}$ which, in the context of atmospheres, is very high. Nitrogen should have long since fled the moon.

- (7) On the ionization of hydrogen, $H \rightarrow p + e^-$. To derive the Saha relation we need to compare the probability of the H gas being in the ionized (or without the electron) state versus the un-ionized (or with electron) state (i.e. H atoms). We can use the Gibbs weight $e^{-\beta(\epsilon - \mu N)}$, for the states. The probability of the ion state (“p” in the book’s notation) when $N = 0$ relative to the atom state (“H”) when $N = 1$ is

$$\frac{P_p}{P_H} = \frac{e^{-\beta(\epsilon_p - \mu \cdot 0)}}{e^{-\beta(\epsilon_H - \mu \cdot 1)}} = e^{-\beta(\epsilon_p - \epsilon_H + \mu)}$$

where ϵ_p is the energy of no electron state and ϵ_H is the energy of the occupied state. Now $\epsilon_p - \epsilon_H = \epsilon$, the ionization energy of 13.6 eV. Thus,

$$\frac{P_u}{P_o} = e^{\beta(-\epsilon - \mu)}.$$

Following the suggestion in the problem,

$$\mu = -kT \ln \left(\frac{V}{N \ell_Q^3} \right) = -kT \ln \left(\frac{2kT}{P \ell_Q^3} \right)$$

where in the second step I used the ideal gas law, neglecting the two degenerate states of the free electron. So

$$e^{-\beta\mu} = \frac{kT}{P \ell_Q^3}$$

and therefore

$$\frac{P_p}{P_H} = e^{\beta(-\epsilon + \mu)} = \frac{kT}{P} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\beta\epsilon},$$

which is equivalent to the Saha relation in equation (5.130). The relation gives the relative abundance of ions to atoms. For example as Schroeder states in Ch 5, on the surface of the Sun this ratio is 8×10^{-5} so less than one hydrogen atom in 10000 is ionized.

- (8) (2 pts.) Black holes continued
 (a) The one particle partition function can be expressed as

$$Z_1 = \frac{2 - e^{-\beta\epsilon}}{(e^{\beta\epsilon} - 1)^2}.$$

The partition function for the whole system is $Z_N = (Z_1^N)/N!$ (as long as the geometric particle are not too “dense”). The average energy is given by the log derivative of the partition function so since $\ln Z = N \ln Z_1 + \text{Junk}$,

$$U = -\frac{\partial}{\partial \beta} \ln Z = N\epsilon \left(\frac{1 - 3e^{\beta\epsilon} + 4e^{4\beta\epsilon}}{1 - 3e^{\beta\epsilon} + 2e^{2\beta\epsilon}} \right).$$

Finally, the heat capacity is then

$$C_v = \frac{\partial U}{\partial T} = \text{ah, well, a bit of a messy function -}$$

see the Mathematica code for this. The code includes plots of dimensionless heat capacity C_v/Nk as a function of dimensionless temperature kT/ϵ . The heat capacity asymptotes to 2 indicating that the degrees of freedom for these particles is 2.

(* 1 particle partition function for bh's *)

In[152]:=

$$Z1 = (2 - \text{Exp}[-\beta * \epsilon]) / (\text{Exp}[\beta * \epsilon] - 1) ^ 2$$

Out[152]=

$$\frac{2 - e^{-\beta \epsilon}}{(-1 + e^{\beta \epsilon})^2}$$

In[153]:=

$$U = - (1 / Z1) * D[Z1, \beta]$$

Out[153]=

$$-\frac{(-1 + e^{\beta \epsilon})^2 \left(-\frac{2 e^{\beta \epsilon} (2 - e^{-\beta \epsilon}) \epsilon}{(-1 + e^{\beta \epsilon})^3} + \frac{e^{-\beta \epsilon} \epsilon}{(-1 + e^{\beta \epsilon})^2} \right)}{2 - e^{-\beta \epsilon}}$$

In[154]:=

Simplify[U]

Out[154]=

$$\frac{(1 - 3 e^{\beta \epsilon} + 4 e^{2 \beta \epsilon}) \epsilon}{1 - 3 e^{\beta \epsilon} + 2 e^{2 \beta \epsilon}}$$

In[167]:=

$$Cv = \text{Simplify}[D[(U /. \beta \rightarrow 1 / T), T] /. T \rightarrow t * \epsilon]$$

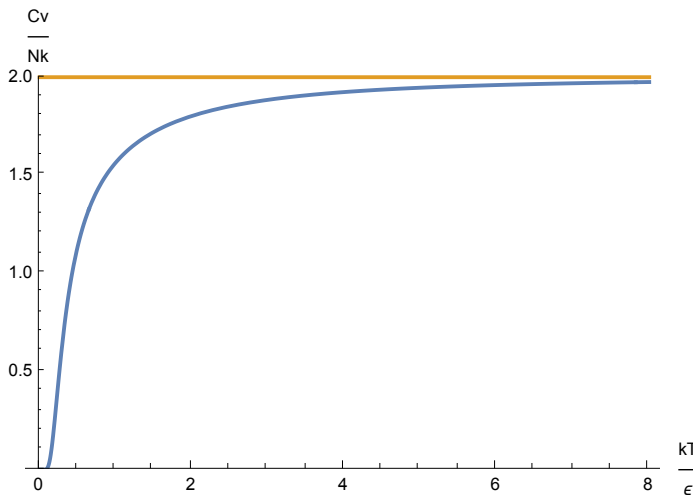
Out[167]=

$$\frac{2 e^{2/t} \left(-2 + 3 e^{\frac{1}{t}} \right)}{\left(1 - 2 e^{\frac{1}{t}} \right)^2 \left(-1 + e^{\frac{1}{t}} \right)^2 t^2}$$

In[200]:=

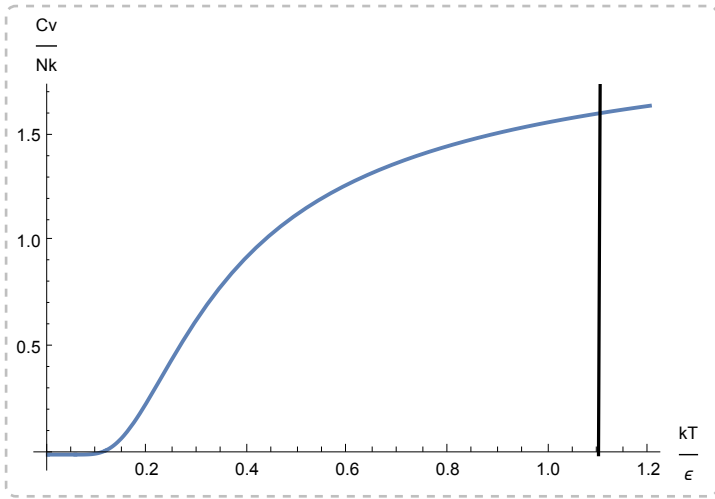
$$\text{Plot}\left[\{Cv, 2\}, \{t, 0, 8\}, \text{PlotRange} \rightarrow \{0, 2\}, \text{AxesLabel} \rightarrow \left\{ \text{HoldForm}\left[\frac{kT}{\epsilon}\right], \text{HoldForm}\left[\frac{Cv}{Nk}\right] \right\}\right]$$

Out[200]=



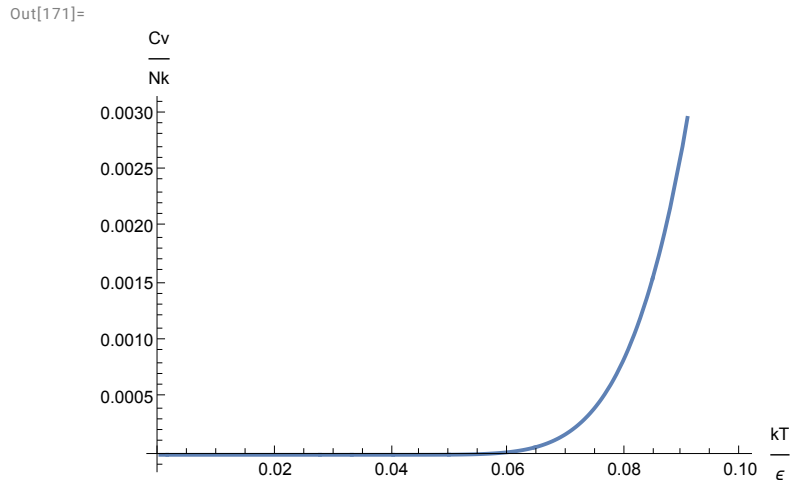
(* Cv has a steep increase starting at dimensionless temperature of about 0.1 and asymptotes to 2 *)

```
In[204]:= Plot[Cv, {t, 0, 1.2}, AxesLabel -> {HoldForm[ $\frac{kT}{\epsilon}$ ], HoldForm[ $\frac{Cv}{Nk}$ ]}]
```



(* I have added a vertical line at about the dimensionless temperature, kT/ϵ of the observers near the horizon *)

```
In[171]:= Plot[Cv, {t, 0, .1}, AxesLabel -> {HoldForm[ $\frac{kT}{\epsilon}$ ], HoldForm[ $\frac{Cv}{Nk}$ ]}]
```



(* Cv is extremely flat at low temperatures! *)

```
In[201]:= N[Pi * .273]
```

```
Out[201]= 0.857655
```

```
In[203]:= N[1 / (Pi * .273)]
```

```
Out[203]= 1.16597
```

- (b) As we saw in class, in the near the horizon limit

$$E_g = \frac{gc^2 A}{8\pi G}$$

Since the energy and area are proportional, the expectation values must be as well,

$$U = \langle E \rangle = \frac{g \langle A \rangle}{8\pi} \text{ or } A_H = \frac{8\pi G}{gc^2} \langle E \rangle.$$

Here, $\langle A \rangle$ is the expectation value for the area of the surface where the observers are, roughly the area of the horizon A_H is this near horizon limit. You'll notice this is the same sort of proportionality as in the paramagnetic problem - see 4(c).

One can also show this by computation: In this non-interacting gas of N particles, the expectation of the area is N times $\langle A \rangle = N \langle A \rangle_1$. In the large-ish j -limit the single particle expectation value is

$$\langle A \rangle_1 = \frac{1}{Z_1} \sum_m 4\pi \ell^2 m^2 e^{\beta \epsilon m},$$

where $m = 2j + 1$. But this expression is proportional to the expectation value of the energy,

$$\langle A \rangle = \frac{4\pi \ell^2}{\epsilon} \left(-\frac{\partial \ln Z_1}{\partial \beta} \right) = \frac{4\pi \ell^2}{\epsilon} \langle E \rangle = \frac{8\pi G}{gc^2} \langle E \rangle,$$

as expected. I used $\epsilon = gc^2 \ell^2 / (2G)$ where $\ell^2 = \hbar G / c^3$ (or, $\ell^2 = \hbar \gamma G / c^3$ if you include the Barbero-Immirzi parameter γ).

- (c) Using

$$S = -\frac{\partial F}{\partial T} \text{ and } F = -kT \ln Z$$

we have

$$S = \frac{\partial \beta}{\partial T} \frac{\partial}{\partial \beta} \ln Z + k \ln Z = \frac{1}{T} \langle E \rangle + k \ln Z$$

or

$$\frac{S}{k} = \beta \langle E \rangle + \ln Z.$$

Using the result from above and the inverse temperature for these observers,

$$\beta = \frac{2\pi c}{\hbar g},$$

we have

$$\frac{S}{k} = \left(\frac{2\pi c}{\hbar g} \right) \left(\frac{gc^2}{8\pi g} \right) \langle A \rangle + \ln Z = \frac{A}{4\ell_P^2} + \ln Z.$$

- (d) Yes!! :-) The last expression has the Bekenstein-Hawking result in the first term (keeping in mind $\langle A \rangle = A_H$)! The acceleration of the observers drops out so this relation holds for all g (in this near horizon limit). Maybe the log term is a correction to the BH entropy...?
- (e) Let's study the log correction term with more care. You could have looked at either the distinguishable or indistinguishable case. I'll discuss both. In the distinguishable case

$$Z_d = (Z_1)^N$$

and

$$\ln Z_d = N \ln Z_1 = \frac{\langle A \rangle}{\langle A \rangle_1} \ln Z_1$$

which could be very large. However at the quoted value for $\beta \epsilon$ the log of the single particle partition function vanishes so we get the Bekenstein-Hawking entropy. This effectively sets

the Barbero-Immirzi parameter γ , a constant in the theory, and is a way to see the current state of affairs in the LQG literature. Here, $\gamma \simeq 0.258$.

In the indistinguishable case, it is more complicated. Using Stirling's approximation

$$\ln Z_i = \ln \left(\frac{Z_1^N}{N!} \right) \simeq \ln \left(\frac{Z_1^N}{N^N e^{-N}} \right) = N (\ln Z_1 - \ln N + 1) = N \ln \left(\frac{Z_1 e}{N} \right).$$

But as we saw above, $N = \langle A \rangle / \langle A \rangle_1$. Hence,

$$\ln Z_i \simeq \frac{\langle A \rangle}{\langle A \rangle_1} \left(\ln Z_1 - \ln \frac{\langle A \rangle}{\langle A \rangle_1} + 1 \right).$$

At the quoted value for $\beta\epsilon$, $\ln Z_1$ vanishes. For a large black hole 1 will be smaller than the log so that

$$\frac{S}{k} \simeq \frac{A}{4\ell_P^2} - \frac{A}{\langle A \rangle_1} \ln \frac{A}{\langle A \rangle_1}$$

in the near horizon, large black hole, large-ish j limit. So there does appear to be a correction term. (A closer look shows that $\langle A \rangle_1 \simeq 11 \ell_P^2$ and so for large black holes the "correction term" may dominate. To be explored later...)