Topics in Mathematical Physics (PHYS 320): QPS 3 Spring 2019

Welcome to the problem set on special functions, Sturm-Liouville theory, and PDEs!

- Please submit your solutions in class on Tuesday April 16.
- Please use your notes Mathematica, Wolfram Alpha, Schaum's, and Boas, but no other resources. If you use software then include printouts of your work using the program(s).
- You may not consult any other resources such as one might find on the internet.
- Your solutions must be entirely your own work.
- Please check your results.

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(1) (20 pts.) Potentials

(a) The electrostatic potential for a charge distribution is the sum of the potentials for each point charge

$$V(r) = \frac{kq}{r}$$

where k is a constant and r is the distance between where the charge and where you are evaluating the potential. Three charges are positioned along the z axis: q at $z = -a$, $-2q$ at $z = 0$ (the origin) and q at $z = a$. Write down the electrostatic potential at an arbitrary point (x, y, z) for this configuration. Write your answer for $V(x, y, z)$ in terms of these coordinates.

(b) Using the law of cosines express this in terms of *spherical* coordinates r and θ where the origin is placed at the $-2q$ charge.

(c) Expand your expression for the potential in terms of Legendre polynominals using the generating function. This is a multipole expansion. To set this up, it may help you to think about studying the field when you are far away, when a/r is small. Simplify your result as much as you can.

(2) (20 pts.) Suppose that you buy an older house with an unheated basement. Hot water (at 125° F) leaves your hot water heater and travels through cylindrical pipes and to your sink. After washing dishes, the water in the pipe is uniformly at 125° F. Find the temperature $T(r, t)$ of the water after it stops flowing, at $t = 0$. Assume that the basement is at 55° F, the water doesn't flow, the pipe is long, and that the pipe material itself is of negligible thickness. The thermal diffusibility ("α^2" in the notation of Boas) of water at these temperatures is 1.43×10^{-7} m^2/s and the pipe has a 1.3 cm diameter. How long do you have before the "hot" water that comes out of your faucet is "cold" - say 57° F - again?

(3) (15 pts.)

(a) Show that the general second order linear differential operator

$$\hat{L}_o = p_o(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x)$$

can be made formally self-adjoint. You can answer this by showing how this is done. You can also use a known solution and demonstrate that it works. Call the new operator $\hat{L}$.

(b) Show that the solutions to the differential equations

$$\hat{L}_o u(x) = 0 \text{ and } \hat{L} u(x) = 0$$

have the same solutions.

(c) When $\hat{L}_o$ is made self-adjoint what happens to the eigenvalue problem

$$\hat{L}_o u(x) = \lambda u(x)?$$

(Congratulations! You have found the "weight function"!)

(d) Do you find the weight function in the inner product for Hermite polynomials (consult your fun facts)?

(e) If you have a solution to Hermite's equation (see the fun fact sheet for the ODE) , $u(x)$, and define a new function

$$\varphi(x) = e^{-x^2/2} u(x)$$

then what is the differential equation for φ? Is this new operator self-adjoint?