General Relativity (PHYS 325) PS 10

Spring 2026 v1.0

These problems are mostly on the orbits in Schwarzschild spacetime. This week in class we'll explore some methods of understanding spacetimes before moving into gravitational waves.

Reading:

- Schutz Chapter 11 on BH's

Problems:

(1) 11.24 (2 pts.) Ignore the extra term of dr^2 in equation (11.127). It is a typo.

(2) Derive the radius of the ISCO for massive particles in the Schwarzschild spacetime.

(3) A stationary observer maintains a fixed radius $R > 2M$ outside a Schwarzschild black hole.
  (a) Find the observer's 4-velocity U_{obs} and express it in terms of the time-like Killing vector ξ^α.
  (b) The observer's companion, a space-faring whale, wishes to escape the black hole on a radial worldline. Let's find the whale's escape velocity, which is defined by the velocity required to reach infinity with zero velocity. Explain why $e = 1$ and $\ell = 0$.
  (c) First show that the whale's 4-velocity at radius R is

$$U^\alpha \to ((1 - 2M/R)^{-1}, \sqrt{2M/R}, 0, 0).$$

  (d) Show that the required energy of the space-faring whale with 4-momentum p^α in the observer's frame is

$$E = -p \cdot U_{obs} = m\left(1 - \frac{2M}{r}\right)^{-1/2}$$

  (e) Thus show that, coincidentally, the escape velocity in this frame, from $E = m/\sqrt{1 - v^2}$, is the same as the Newtonian case

$$v = \sqrt{\frac{2M}{R}}.$$

(4) Roger falls radially into a black hole:
  (a) In class we found a conserved quantity e for BH orbits and trajectories. If Roger starts from rest relative to a stationary observer Titty at $r = 10M$ then show that the value of this conserved quantity e for the observer is $2/\sqrt{5}$.
  (b) How much proper time elapses on Roger's worldline before hitting the singularity?
  (c) How long (in seconds) would this be for a 10 solar mass black hole?
  (The characters in this problem are the younger members of the Walker family in Arthur Ransome's "Swallows and Amazons".)

(5) (2 pts.) Recent results from quantum gravity: Lewandowski et. al. suggest that the metric for spherically symmetric spacetimes is[1]

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{\beta M^2}{r^4}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{\beta M^2}{r^4}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

[1] The preprint for this paper is 2404.01418 on the arXiv.

where the small $\beta \simeq \frac{G\hbar}{c^3} \simeq 10^{-66}$ cm^2.

(a) Find the Killing vectors for this metric.

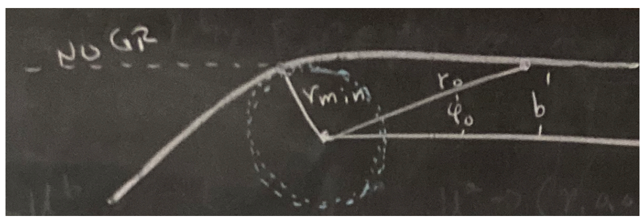
(b) Derive the new geodesic equation for light in the equatorial plane. Sketch or plot the effective potential. Discuss the orbits.

(c) If

$$f(r) = \left(1 - \frac{2M}{r} + \frac{\beta M^2}{r^4}\right)$$

has real roots, what is the minimum mass of such a black hole? Work this out in both the above units and as compared to the mass of a rain drop just about ready to fall from a cloud. Assume a spherical drop of water with diameter 200 μm.

(6) (Optional 1 pt) This problem completes the calculation of the deflection of light around the sun. Here's a sketch:



(a) Starting with our result for the effective potential for light, show that with the change of variables $u = b/r$ the "effective energy conservation" equation for light becomes

$$\left(\frac{du}{d\varphi}\right)^2 = 1 - u^2\left(1 - \frac{2M}{b}u\right).$$

This is for orbits in the equatorial plan.

(b) Using the mass and radius of the sun, $R = 6.96 \times 10^5$ km, show that the quantity $2M/b$ is always small.

(c) Given this, we'll keep only the first order result in M/b for this calculation. Let

$$y = u\left(1 - \frac{Mu}{b}\right)$$

and show that

$$\frac{d\varphi}{dy} \simeq \frac{1 + \frac{2M}{b}y}{\sqrt{1 - y^2}}, \tag{1}$$

to first order in $2M/b$.

(d) The total deflection $\Delta\varphi$ is twice the angle of deflection when integrating from $r \to \infty$ to $r = r_{min}$. We can integrate equation (1) to obtain the angle. Notice that (now) the integration in y separates into two relatively simple integrals. Trace through the change of variables to obtain the limits. Compute these to obtain the result

$$\Delta\varphi \simeq \pi + \frac{4M}{b}$$

(e) Assuming the null geodesic just grazes the surface of the sun find the deflection angle $\delta\phi = \Delta\varphi - \pi$.