

Before break we found Einstein’s equations “curvature = stuff” from curved spacetimes, the equivalence principle, energy conservation, and the Newtonian limit. Now we can solve for a bunch of different, physically relevant cases. We’re starting with the Schwarzschild black hole and the Newtonian limit.

**Reading:**

- Schutz Chapter 7 pages 170-172 (we skip the next bit on fluids), sections 2 - 4
- Schutz Chapter 8 sections 1 - 2
- Schutz Chapter 10 sections 1 - 2

**Problems:**

All numbered problems are from Schutz.

- (1) *Newtonian physics from GR* Before break you found the Christoffel symbols for the metric

$$ds^2 = -(1 + 2\phi)dt^2 + (1 - 2\phi)dx^i dx_i,$$

where  $\phi$  is some function on spacetime. Assuming  $\phi$  is small compared to 1

$$\begin{aligned} \Gamma_{tt}^t &\simeq \partial_t \phi \\ \Gamma_{tt}^i &\simeq \delta^{ij} \partial_j \phi \\ \Gamma_{it}^i &\simeq \partial_t \phi \\ \Gamma_{ii}^i &\simeq -\partial_i \phi \\ \Gamma_{ii}^t &\simeq -\partial_t \phi \text{ and} \\ \Gamma_{it}^t &\simeq \partial_i \phi \end{aligned} \tag{1}$$

with no sum over  $i$ 's.

- (a) Show that the geodesic equation for 4-velocities implies that

$$p^\alpha \nabla_\alpha p^\beta = 0 \tag{2}$$

where  $p^\alpha$  is the 4-momentum.

- (b) Taking the spatial components of this equation find the equation of motion. Assuming that  $|p^i| \ll p^t$  and the Christoffel symbols derive an approximate form of the equation of motion for (relatively) slow motion. By comparing this to  $d\mathbf{p}/dt = \mathbf{F}$  explain what  $\phi$  is.
- (c) By taking the time-like component of the equation (2), show that the particle’s energy is conserved if  $\phi$  is time-independent.
- (d) **Delayed until later** In the static case,  $\phi = \phi(x, y, z)$ , find  $R_{iit}^t$  (no sum over  $i$  here) in the limit of small  $\phi$ .

- (2) (worth 4 points) The beginning of the analysis of curvature quantities for “our metric”

$$(g_{\alpha\beta}) \rightarrow \begin{pmatrix} -e^{2\phi} & 0 & 0 & 0 \\ 0 & e^{2\lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \tag{3}$$

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Note that  $\phi = \phi(r)$  is a function of  $r$  in the metric and  $\varphi$  is the coordinate. Both  $\phi$  and  $\lambda$  are functions of  $r$  only. Let's denote the derivatives with primes so that, for instance,

$$\frac{d\phi}{dr} = \phi'$$

- (a) Find the inverse metric and then the Christoffel symbols. Show all your work. So you can check your work here's a couple Christoffels

$$\Gamma_{tr}^t = \phi' \text{ and } \Gamma_{\theta\theta}^r = -re^{-2\lambda}$$

- (b) Now compute at least three of the non-vanishing components of Riemann. Show all your work. Here's a couple of components to get you started,

$$R_{rtr}^t = -\phi'' + \lambda'\phi' - (\phi')^2 \text{ and } R_{\varphi\theta\varphi}^\theta = \sin^2\theta(1 - e^{-2\lambda}).$$

Feel free to use Mathematica to check your results but be sure to have complete work for three non-vanishing components.

- (c) Find the non-vanishing components of the Ricci tensor  $R_{\alpha\beta}$ , the Ricci scalar  $R$ , and finally the Einstein tensor  $G_{\alpha\beta}$ . You may use Mathematica for this part.
- (3) (2 pts.) One of the best ways to identify spacetime symmetries and the associated conserved quantities is through Killing vectors (so named for Wilhelm Killing, a German mathematician) and one of the easiest ways of identifying these vectors is by asking - does the metric depend on each coordinate? For example, *if the metric doesn't depend on the coordinate  $x^3$*  then the transformation

$$x^3 \rightarrow x^{3'} = x^3 + \epsilon$$

(or any constant) leaves the metric unchanged. The vector

$$\xi^\alpha \rightarrow (0, 0, 0, 1)$$

is in the direction of the transformation. As you can see,

$$x^{\alpha'} = x^\alpha + \epsilon \xi^\alpha \text{ or } x^{3'} = x^3 + \epsilon$$

for an infinitesimal shift by  $\epsilon$ . This vector  $\xi^\alpha$  is a Killing vector for the translational symmetry of  $x^3$ .

- (a) Find the 3 similar translational symmetries of the Minkowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

There are 6 more symmetries. Can you identify them?

- (b) Killing vectors satisfy the equation

$$\nabla_{(\alpha}\xi_{\beta)} = \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha = 0.$$

To see why it is useful, show that if the vector field  $\xi$  satisfies the above equation then the quantity  $p^\alpha\xi_\alpha$  is constant along a geodesic, thus identifying a conserved quantity! Start this problem by taking the directional derivative  $U^\mu\nabla_\mu$  along a geodesic.

Since this holds for each Killing vector, there is an associated conserved quantity for each symmetry.

- (4) An infinitesimal Lorentz transformation with speed  $v$  in the  $x$  direction can be written as

$$\begin{aligned} x'^0 &= x^0 - vx^1 \\ x'^1 &= x^1 - vx^0 \end{aligned} \tag{4}$$

- (a) Show that the Killing vector  $\xi^\alpha \rightarrow (x, t, 0, 0)$  corresponds to this infinitesimal transformation.

- (b) Sketch  $\xi^\alpha$  in a few places in the  $t - x$  plane, in the quadrant  $x > 0$  and  $x > |t|$ .
- (c) Show that this Killing vector satisfies Killing's equation.